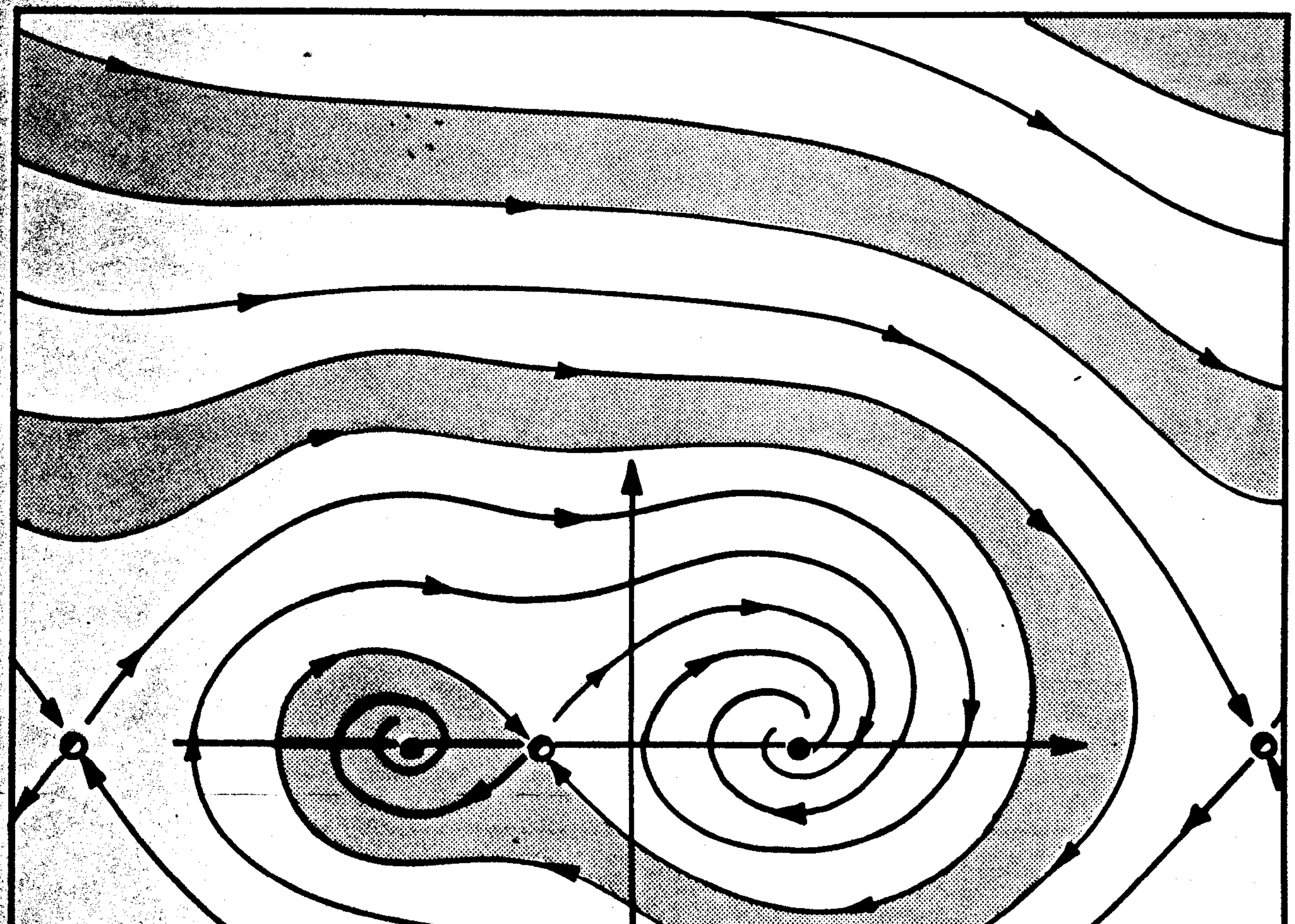
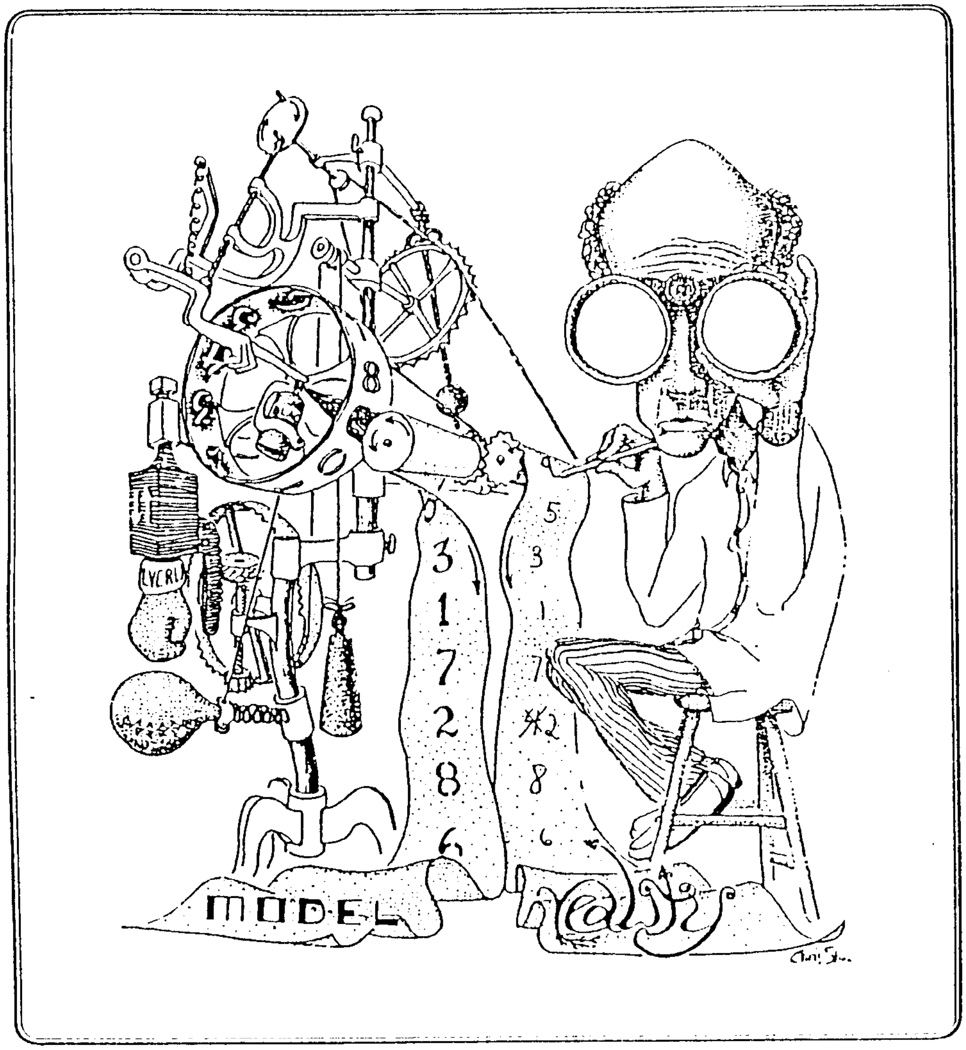
The Science Frontier Express Series

DYNAMICAL SYSTEMS
A Visual Introduction



Abraham, Abraham & Shaw

Aerial Press 1990



THIS MODELER WANTS YOU

From R. Shaw *The Dripping Faucet as a Model Chaotic System*

INTRODUCTION

Some Attractive Features of the Dynamical Modeling Strategy

There is a revolutionary new strategy of mathematical modeling of systems called dynamical systems theory. While its roots reach back to Newton, Rayleigh, and Poincaré, the past two decades have witnessed a revolution in its language, concepts, and techniques for dealing with complex cooperative systems evolving through multiple modes of dynamical equilibrium (static, oscillatory, and chaotic). Hence, their applicability to all scientific disciplines which have increasingly in the same period come to be understood in a similar light of synergistic cooperative systems, from quantum physics and cosmology, to biology, psychology, and sociology. The mathematics provides models, simulation, cognitive strategies, and intuitively clear geometric representations for complex systems. It also serves as a unified philosophic view for integrative, hierarchically organized systems, and for dissipative, irreversible, evolutionary dynamics. In short, it is a world view as well as an elegantly simple modeling strategy. It is emerging as the metalanguage, the *metaparadigm*, of science. Several essential features contribute to this developing hegemony.

The *first* is that instead of looking at a static object or event as a thing to be explained, it looks to a set of complex evolving relationships as both the subject, the object, and the explanation, *holistic* and dynamical.

A *second* feature is that nonlinear dynamics takes an essential set of rules (e.g., equations, usually ordinary differential equations) defining those relationships and shows how very different types of organization in those relationships can emerge from those same rules, as a critical feature (control parameter) changes. These changes in system organization or performance are called bifurcations. That is, the evolution of a system can be gradual most of the time, but be punctuated by these dramatic bifurcations as the control parameter passes through critical thresholds (bifurcation points). Thus, *saltatory* or *step-wise evolution is seen as natural*, not magical. It derives from orderly principles; extraordinary events external to the system need not be invoked to explain it.

Third, **causality** is considered multilevel and multideterminate. Any level or domain of observation or theory may communicate with others without necessarily going through a linear causal chain as from microscopic to macroscopic, from subatomic to cosmological or social, from independent to dependent variable, from stimulus to response, admittedly rather outdated concepts anyway. *Dynamical theory is very well disposed to relating several levels of observation.* The language of reductionism and independent and dependent variables gives way to the language of *dynamical interactive variables*. Interactions include the systems concepts of *feedback* and *control*, which lead to the fourth main feature.

This *fourth* feature, **self-organization**, derives from combining the concept of feedback and the concept of the control parameter. This occurs when the *value of the control parameter depends on the state of the system; the system has control over itself.* In psychological and social systems, individuals and societies become aware of critical control parameters and learn to control them for self-improvement (hopefully).

A *fifth* feature is that the visual geometric dynamical approach is highly **communicative**, requiring mathematical maturity, but without requiring a great deal of technical mathematical sophistication. Currently, the various modeling strategies in science have become quite provincial in two senses. The first is that their idiosyncratic and complex languages make communication difficult. The second is that the technical difficulty of any scientific enterprise usually confines it to the study of but a few levels and variables out of the many involved. The dynamical modeling strategy attempts to transcend these limitations without the danger of limiting heterogeneity in science. Computer programs for theoretical simulation, data treatment, and graphic display have simplified the communication process. Dynamical modeling should help to increase *communication among disciplines* and to increase the ability to interrelate findings in different disciplines.

A *sixth* and final critical feature that we will focus on here, is the importance of **chaos**. Dynamical theory describes three main types of temporal organization or attractors: fixed, cyclic, and chaotic. Science has tended to emphasize fixed and cyclic behavior, but recently there has been a growing appreciation of more complex 'chaotic' temporal structure. Chaotic processes transcend the usual tendency of our experimental designs to dispose of them as random Gaussian, Poisson, or other forms of 'error' distributions messing up our experiments. Nonlinear dynamics thus provides us with a perspective for recapturing much that was lost to such conceptual and experimental limitations. It has implications for improved experimental designs emphasizing multiple measurements over time. Studying changes in states over time has long been considered important, but previously we have been impoverished without adequate conceptual, empirical, or linguistic (communicative, computational, or graphic) tools with which to exploit this point of view.

This volume is designed as an introduction both for those considering the use of this approach for their own research and those interested in becoming a bit more fluent in the language and concepts of dynamical theory. The emphasis on visual representation makes the technical ideas intuitively accessible and usable. Symbolic notation and equations of algebra and calculus are kept to a minimum and mostly collected into the optional appendix. While vector calculus provides the foundation of the subject, fluency in calculus is not assumed on the part of the reader, but will be helpful to those wishing to go deeper into the subject. The visual approach can be obtained more completely from Abraham & Shaw (1982-1988), and the mathematical-symbolic approach can be obtained more completely from works cited there, from Thompson & Stewart (1986), or from a good computer simulation program such as *Dynasim* (Abraham, 1979), *Dynamical Software* (Schaffer, Truty, & Fulmer, 1988, or *Chaos Demonstrations* (Sprott & Rowlands, 1990) for personal computers, or the emerging programs available for workstations with outstanding graphic engines (unpublished programs by Stewart and others). As an elementary tutorial, it should be appropriate for those just getting interested in dynamical systems.

This volume represents a condensation of Abraham & Shaw, *Dynamics The Geometry of Behavior,* 1982-1988, and is adapted from there and from Abraham & Shaw (1987) and from Abraham, Abraham, & Shaw, *A Visual Introduction to Dynamical Systems Theory for Psychology*, 1990. All figures are taken from these sources (as per back of title page) unless explicit citation specifies otherwise.

DYNAMICS

A. DEFINITIONS OF ELEMENTARY TERMS

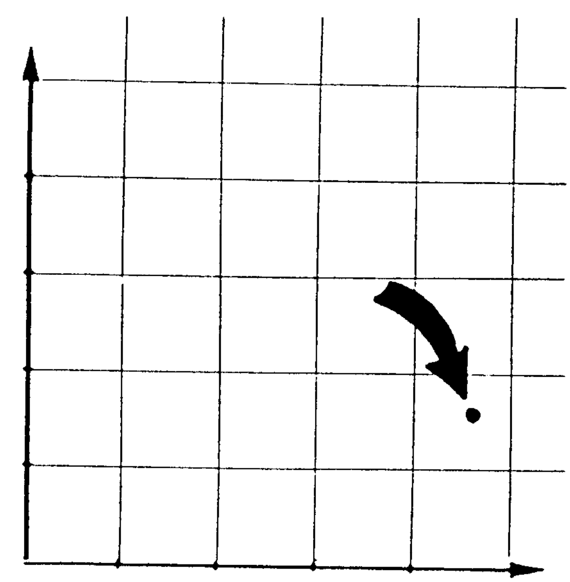
1. STATE SPACES

Most systems investigated in science involve sets of interacting factors. They can be as straightforward as the interaction of position and velocity in a pendulum, or as complex as the interactions within a living cell, its communication with other cells and their environment, and the communications between individuals in social organizations. The process of modeling such systems is familiar enough to us. Some aspects are real (observable variables), while others may be imaginary (intervening or hypothetical variables), and we try to discover as many of these as possible and characterize the relationships between them (MacCorquodale & Meehl, 1948). A system, then, is a set of such variables whose values change over time.

Figure 2

STATE SPACE

The state space is the graphical representation of all the possible states of the system, that is, all values of all the variables under consideration, here shown as a 2D Euclidean plane with one of the observed states indicated.



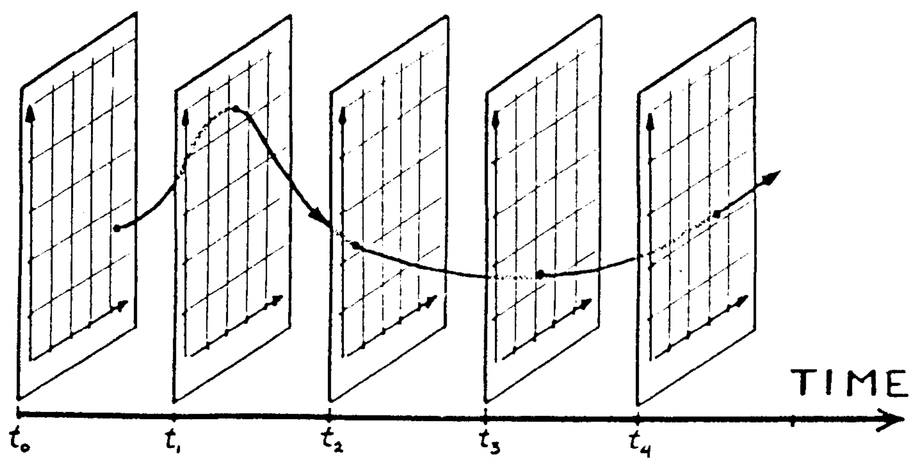
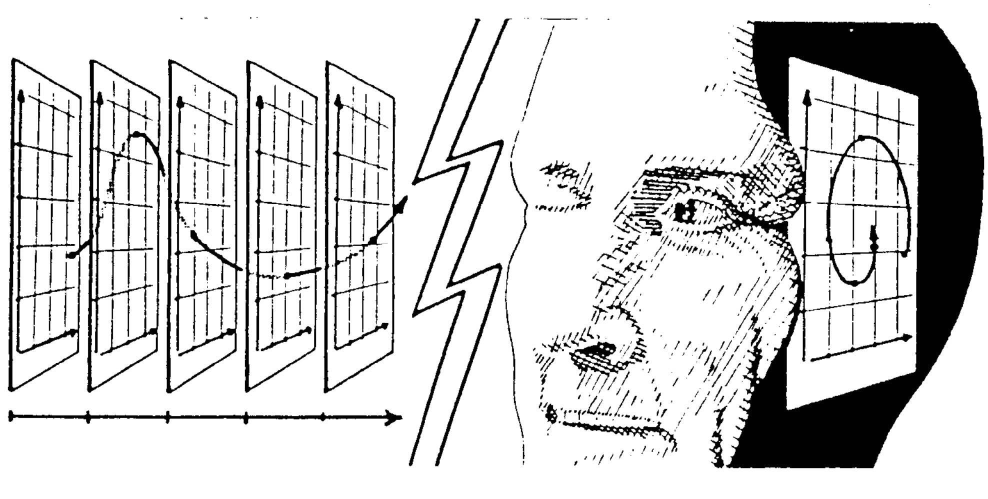


Figure 3. TIME SERIES

The changes in state observed over time may be represented by a conventional time series, the state space shown (vertical planes) as a function of time which is given its own (horizontal) axis.

Figure 4. VIEWING THE TRAJECTORY

The trajectory may be obtained from the time series by viewing straight down the time axis from infinitely far away.



2. TRAJECTORIES AND PHASE PORTRAITS

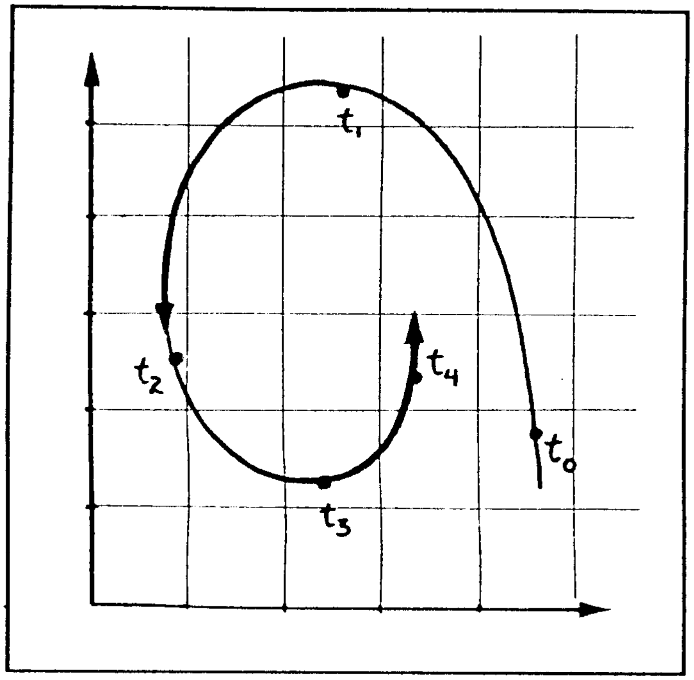


Figure 5

TRAJECTORY

A trajectory shows a history of the states of a system from a given initial state. The time axis is removed, and time is indicated by the use of arrows and time labelling.

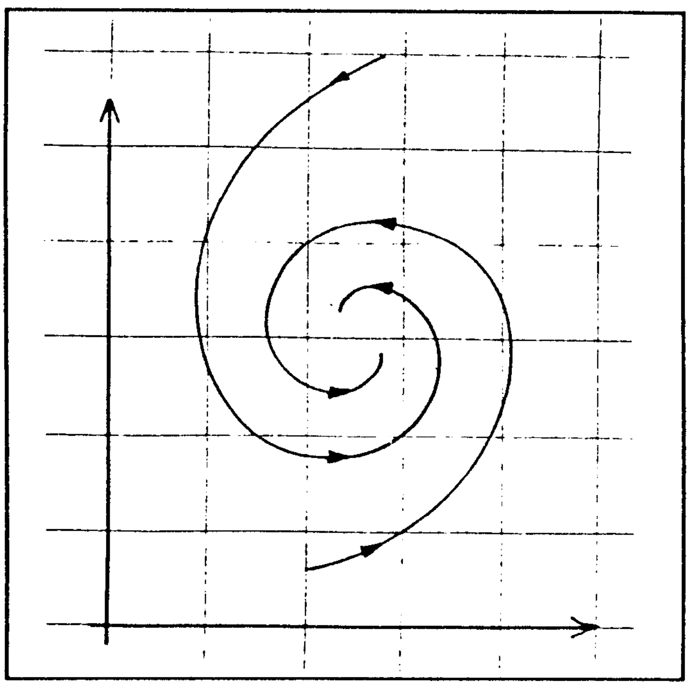


Figure 6

PHASE PORTRAIT

The state space, filled with trajectories generated by a given model for different initial states (only a few representative ones are usually drawn).

3. VECTORFIELDS AND DYNAMICAL SYSTEMS

Dynamical systems are systems with special qualities. To describe these properties it is necessary to introduce some special kinds of vectors. If we take any two points (vectors) on a trajectory, the difference between them is a new **bound vector** (Fig. 7). As mentioned before, the rate of change in each variable is reflected in the distance between points. This rate, or average velocity of the change of state may be represented by an **average velocity vector**, which is the bound vector divided by the interval of time, to yield the rate of change in each variable per unit of time. So now we have an **average velocity vector** representing rate of change of the state variables which can be represented numerically, as

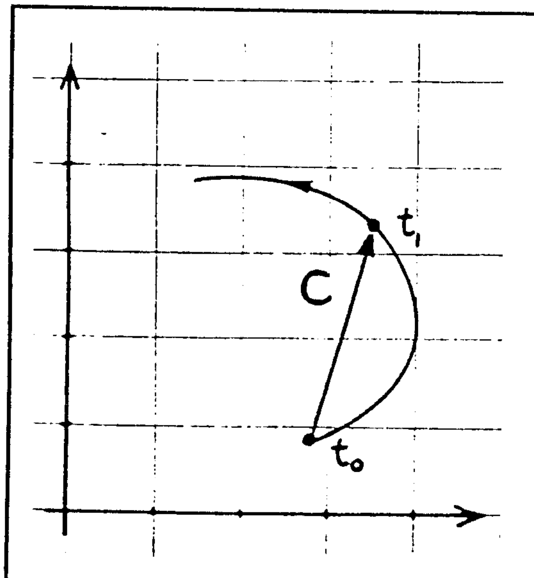


Figure 7

BOUND VECTOR

The states observed at two different times may comprise a bound vector, denoted here by the pointed line segment, C.

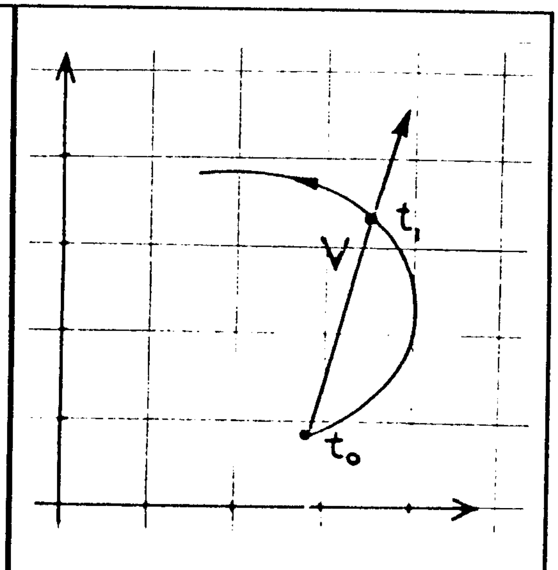


Figure 8

AVERAGE VELOCITY VECTOR

The average velocity vector is the bound vector divided by the elapsed time, T, between t_0 and t_1, $V = C/T$.

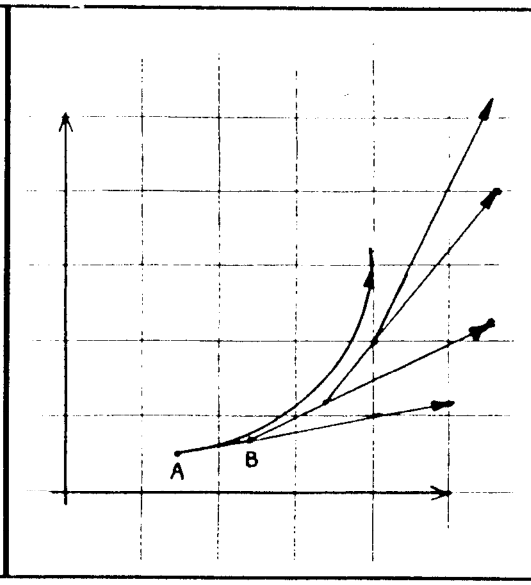
just calculated, or visually in the state space (Fig. 8). The **average velocity vector**, thus, represents the average rate and direction of the change in the state of the system between two points in time.

Now suppose we make our measurements continually in time, or decide our trajectory may be represented by a continuous model. Further, suppose we allow the time for the second point in time to get closer and closer to our first point in time. As this time shrinks, the average velocity vector becomes a **tangent vector** at the first point in time (Fig. 9). It is also called the **instantaneous velocity** of the trajectory at that point in time, representing the instantaneous rate and direction of change in the state of the system at that point in time.

Figure 9

TANGENT VECTOR

The instantaneous velocity,
or tangent vector,
is the limit vector that V tends to
as the elapsed time, T,
shrinks infinitesimally small.



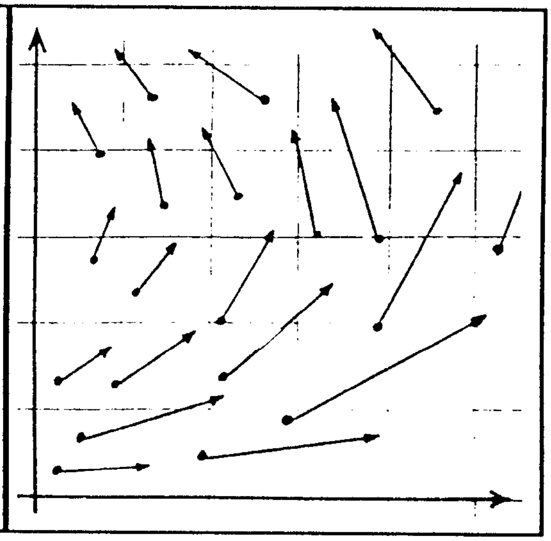
What does this tangent vector tell us? Why have we derived it from the trajectory under the pretense of having a continuously instead of a discretely changing variable? What is the information contained in the instantaneous velocity vector? Simply this. It tells us the tendency of the system to change when in that state. It says in what direction and how fast the system should change on all variables simultaneously. It represents the forces which generate the trajectory. It moves the system to the next point on the trajectory where the next vector governs its next move. This process of deriving this instantaneous velocity vector is known as differentiation in vector calculus (the use of calculus will not be employed here; only the visual or geometric interpretations of dynamical systems are used which are hopefully intuitively clear without an extensive knowledge of calculus).

The modeling process begins with the choice of a particular state space in which to represent observations of a real system. Prolonged observations lead to many trajectories within the state space for differing initial points. At any point on a trajectory, a velocity vector may be drawn. This was the new method of *fluxions* of Newton. This calculus is the foundation of dynamics. It was also developed by Leibniz whose notation proved more accessible. It represents the inevitable culmination of the experimental work of Galileo. and Kepler who inaugurated the modern study of the development of time and motion. The velocity vector describes the inherent tendency of the system to move at each point in the state space. The prescription of a velocity vector at each point in the

Figure 10

VECTORFIELD

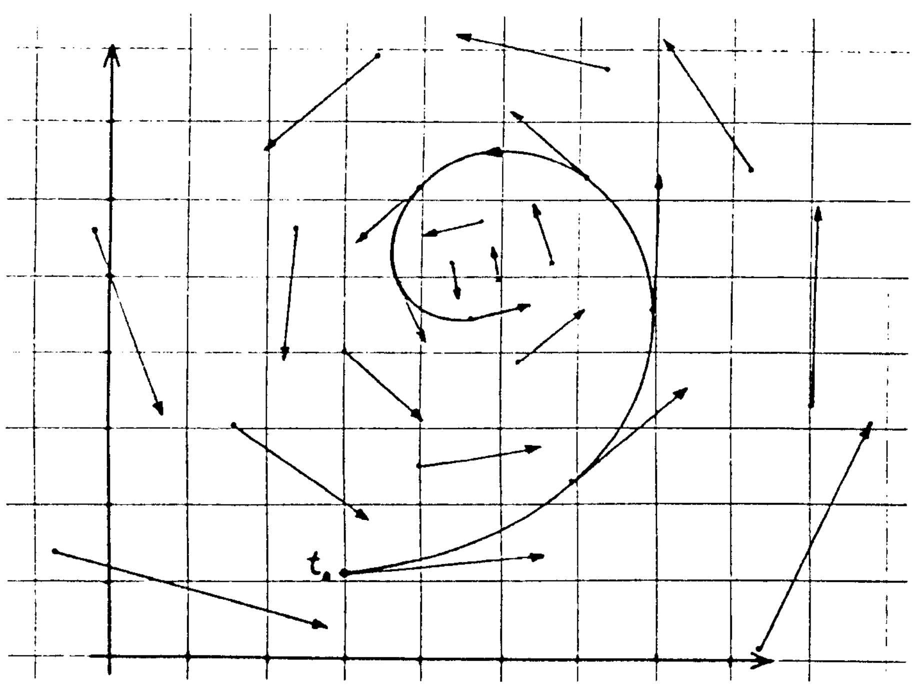
A vecrtorfield is a field of bound vectors, one defined (and bound to) each and every point of the state space.. Usually, as here, only a few are drawn to suggest the full field.



state space is called the **velocity vectorfield** (Figs. 10 & 11). The velocity vectorfield is derived from the phase portrait by differentiation. The phase portrait may be obtained from the dynamical system by integration. The phrase **dynamical system** specifically denotes the vectorfield.

Figure 11. TANGENT VECTORS AGREE WITH VECTORFIELD

A trajectory has a velocity vector at every point along it that agrees with the vectorfield. The trajectory evolves from the initital state, at time t_0, so as to be tangent to the vectorfield at each point.



If there are some regularities to the observed patterns in the phase portrait and the vectorfield, then it may be possible to model the dynamical system by ordinary differential equations. These properties (hypotheses) are:

Hypothesis 1. *The observation of the system over time, represented as a trajectory in the state space, has the property that at each of its points, its velocity vector is exactly the same as the vector specified by the dynamical system.*

Hypothesis 2. *The vectorfield of the model is smooth.*

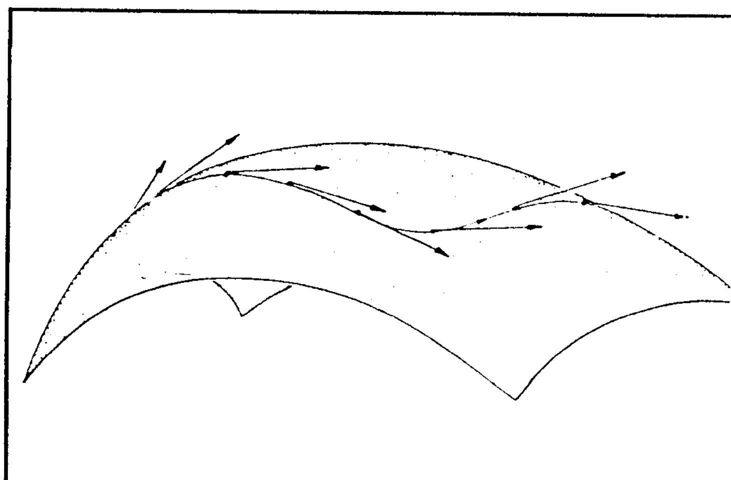


Figure 12

MANIFOLDS

Generalized state spaces, manifolds, may include curved surfaces. Here a trajectory remains on the surface while the tangent vectors project off of it.

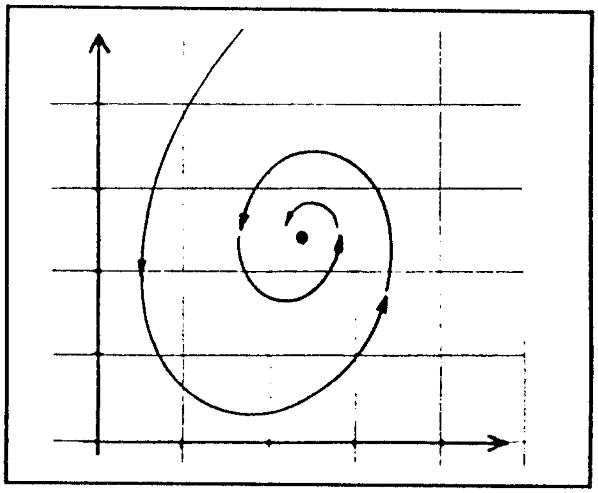
4. ATTRACTORS, BASINS, SEPARATRICES, REPELLORS, SADDLES

Experimental trajectories may present patterns if the systems have any regularity to their behavior. The task of the scientist is to discover those patterns; that of the modeler to discover reasonable models approximating the same patterns. A simple taxonomy of some common patterns can be given. A point with a zero instantaneous velocity vector is a special kind of trajectory called a constant, critical point, rest point or fixed point. If all nearby trajectories tend to that point, it is called a **fixed point attractor** or **static attractor** (Fig. 13). It can represent static equilibrium and homeostasis. The portion of the state space occupied by trajectories approaching the attractor is called the **inset** or **basin** of the attractor. If all nearby trajectories depart the region around the limit point, it is called a static or **point repellor** (Fig. 14). The set of departing trajectories and the state space they occupy is called the **outset** of the limit point.

Figure 13

FIXED POINT ATTRACTOR

All nearby trajectories approach this limit set, the fixed point attractor, just like the one shown. They comprise its inset.

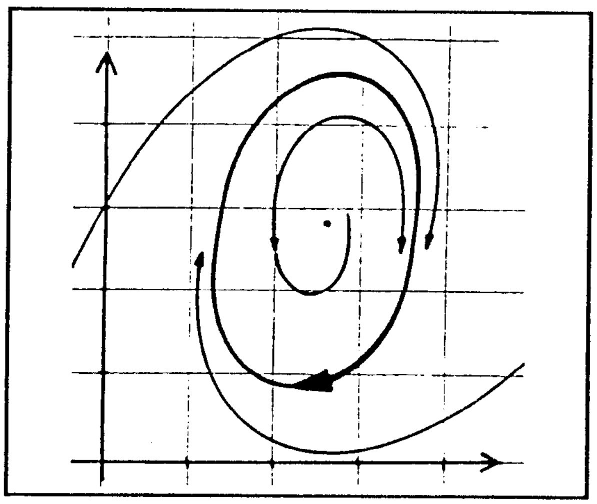


The phase portrait may have a **limit cycle.** If all trajectories approach this limit cycle, it is a **periodic attractor** (also **cyclic attractor**) and represents periodic equilibrium (Fig. 14). Except for the critical point in the center, the point repellor, the trajectory from every single initial state evolves to the periodic attractor. The inset includes the disk inside the attractor and the open annulus outside it. Note how this portrait differs from that of Fig. 15 which also shows a cyclic attractor. Here the point in the center is also an attractor. Between the fixed point attractor and the periodic attractor there is another exceptional limit set, also a cyclic one, a trajectory that tends to neither attractor, but separates the basins of the two attractors. It is a **separatrix.** It is also a **cyclic (periodic) repellor.**

Figure 14

STATIC REPELLOR & CYCLIC ATTRACTOR

The limit point in the center is a repellor. All nearby trajectories depart and apprach the cyclic attractor and comprise its outset. All trajectories from both inside and outside comprise the inset of the cyclic attractor.



The inset of an attractor is its **basin**, that is, it is comprised of trajectories tending to that attractor. The probability that any initial state will approach an attractor is proportional to the volume (or the length or area, depending on the dimensionality of the state space) of its basin. Phase portraits will often have more than one basin (Figs. 15 & 16A). Any points or trajectories not in any basin (i.e., not tending toward any attractor), by definition belong to a separatrix. If the separatrix separates basins, as in Figs. 15 & 16A, it is an **actual separatrix**, else it is a **virtual separatrix**.

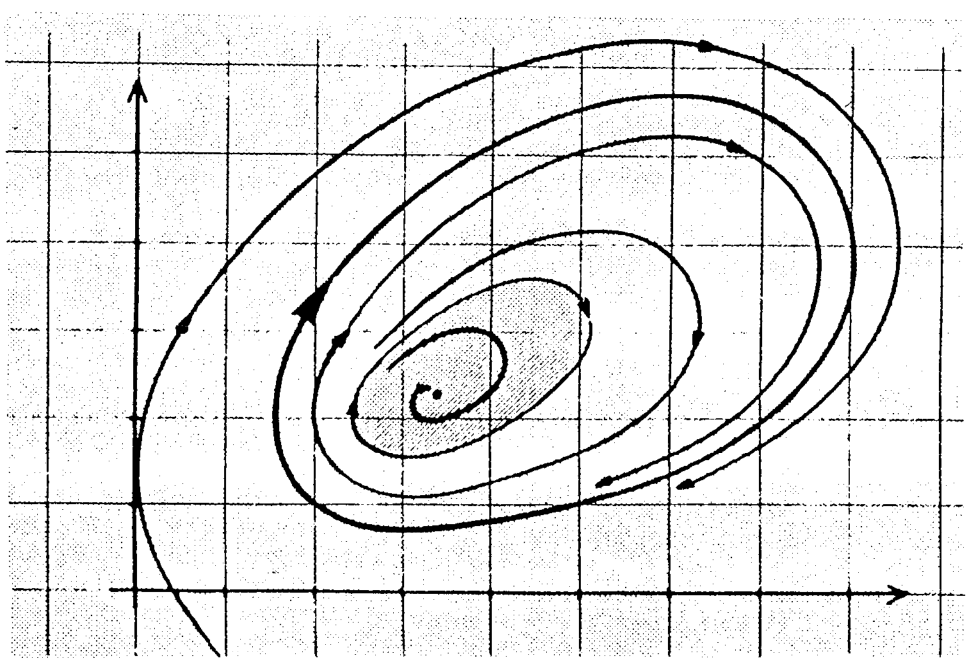


Figure 15. BASINS & SEPARATRIX

This is a phase portrait with a periodic repellor as a separatrix between the basins for the cyclic attractor and the fixed point attractor inside.

There is another type of limit set that is neither an attractor nor a repellor; rather, it has both properties. These are called **saddles**. Fig. 16A shows a saddle point flanked by two fixed point attractors. Insets to saddles are separatrices. In this case, they comprise a virtual separatrix between two basins. Unlike the stable fixed point attractors, fixed point repellors and saddles points are inherently unstable. Homeostasis is a fixed point attractor. A pin on its point falls with the slightest perturbation. The rider falls back into the saddle when perturbed antero-

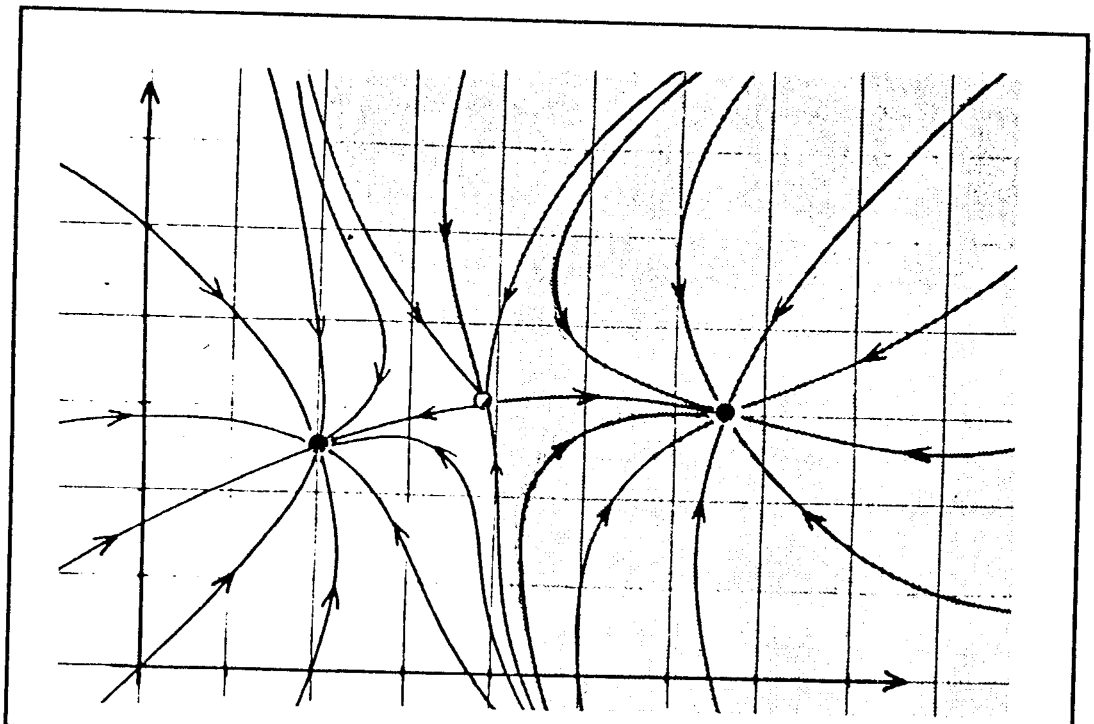


Figure 16A. SADDLE POINT

This is a phase portrait with two fixed point attractors. The third point is a **saddle**, not an attractor: some nearby trajectories approach it, but others depart. The insets to this **saddle point** comprise the separatrix. The outsets of the saddle belong to the insets of the point attractors.

posteriorly along the saddle inset and falls onto the ground when perturbed laterally along the outset of the saddle.

Insets of saddles (exceptional limit sets) are separatrices, since they are not tending to attractors. Suppose a saddle point flanked, as in Fig. 16B, by a repellor and an attractor (or another saddle), out of view to the southwest, and assume that except for the inset to the saddle, all nearby trajectories tend to that southwest limit set. Then note that the insets of the saddle do not separate basins and thus comprise a virtual separatrix. (Test yourself by comparing right and left panels of Fig. 80 as to which possess actual versus virtual separatrices).

Chaotic attractors are attractors that are neither fixed nor cyclic. Discussion of them is deferred. For now, winter arrives, and our equestrienne/-an becomes an alpine skier.

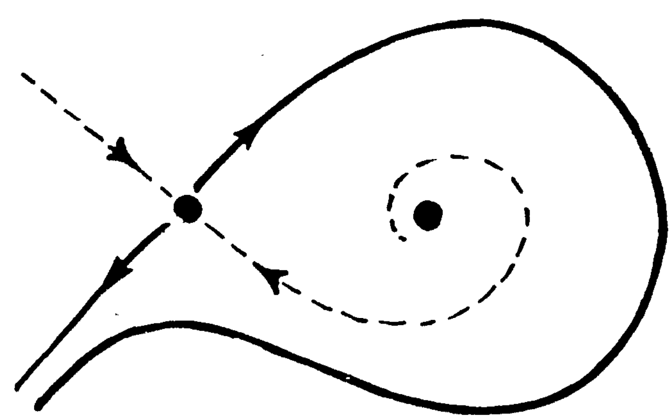


Figure 16B. VIRTUAL SEPARATRIX

Insets to a saddle that fail to separate basins because all nearby trajectories tend toward a saddle or attractor off to the southwest, constitute a virtual separatrix.

5. GRADIENT SYSTEMS

The *gradient* operation of vector calculus provides dynamical systems of an especially simple type called a *gradient system*. These systems have an auxiliary function, called the *potential function*. The velocity vectorfield is simply the gradient vectorfield of this potential function.

Figure 17

THE POTENTIAL FUNCTION

The state space, in this example, is the plane. The potential function is a function from the state space to the real number line. To each point in the state space, it assigns a real number. This number is the potential of the corresponding state.

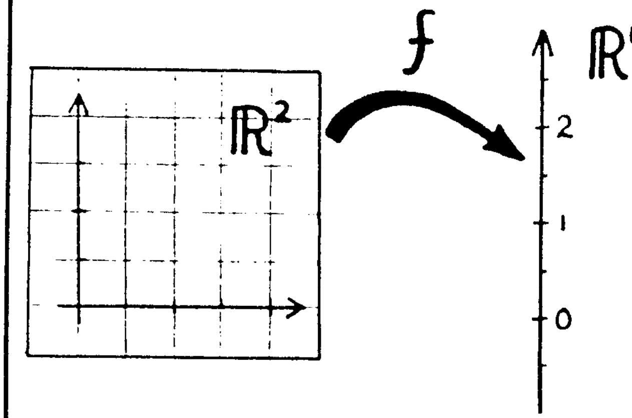
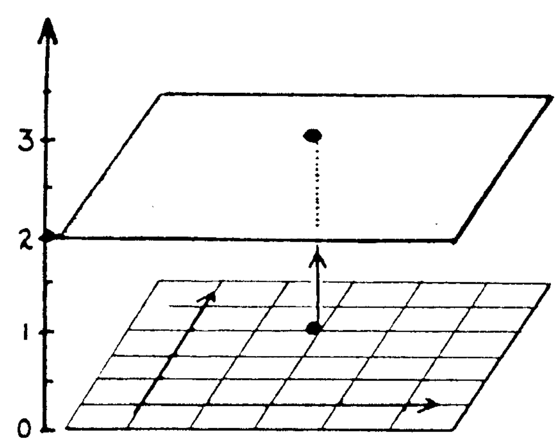


FIGURE 18

3D REPRESENTATION OF THE POTENTIAL AS A FUNCTION OF THE 2D STATE SPACE

From each state in the horizontal plane of the state space, move vertically a distance equal to the potential of that point.



The graph of a potential function on a planar state space is a surface in three-dimensional space, called the potential surface. We may think of this as a landscape.

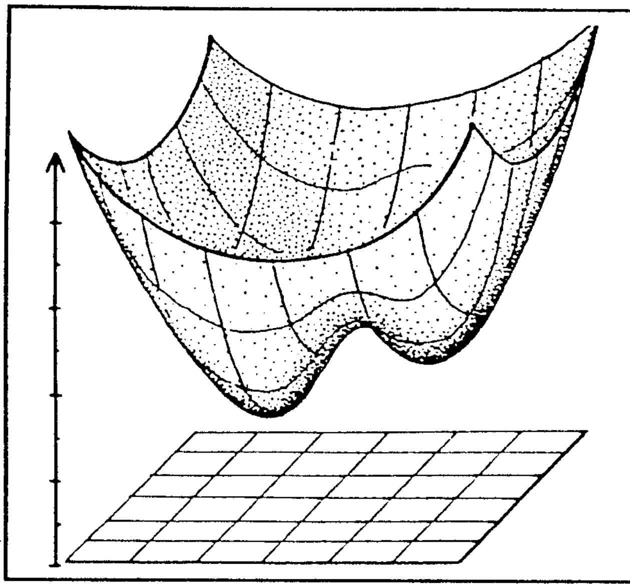


Figure 19

THE POTENTIAL SURFACE

The potential function may be visualized as a potential surface, here shown for some particular example with two valleys and a saddle ridge in between.

An alternate representation of a function is its countour map. Start like this.

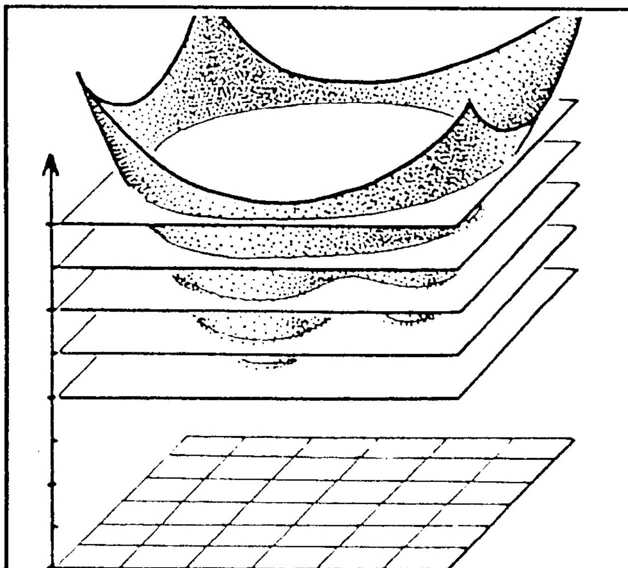


FIGURE 20

LEVEL CURVES

The intersection of horizontal planes with the potential surface create level curves on the surface.

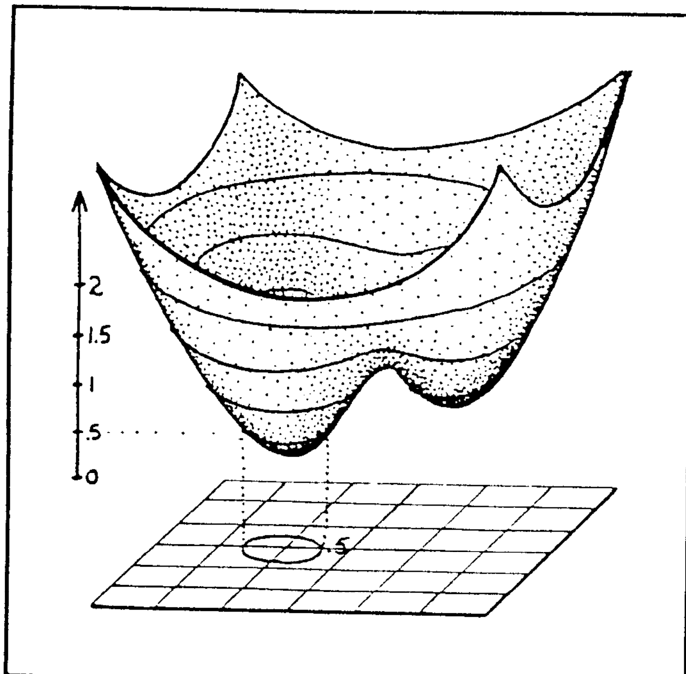


Figure 21

CONTOUR CURVE

Next, project each level curve down onto the plane of the state space. This curve in the state space, also called a contour curve, contains every state with the same value of the potential function. This curve may be labelled with its common value of the potential.

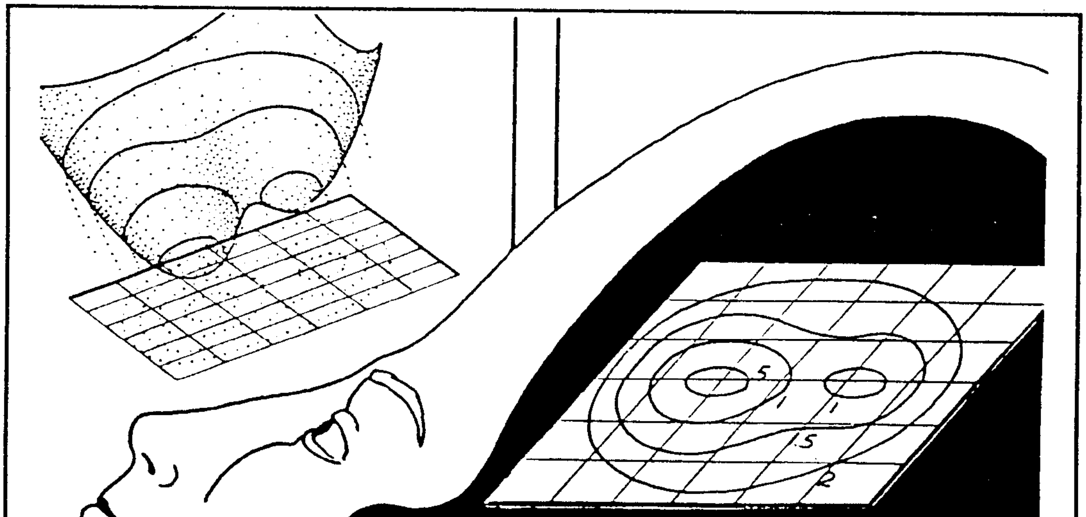


Figure 22. CONTOUR MAP

Repeat for each of the level curves yielding the contour map of the potential surface on the horizontal plane, here viewed from below.

Finally, remove the contour map from the three dimensional context. This is an alternative representation.

The gradient dynamical system for this particular potential function is derived as follows.

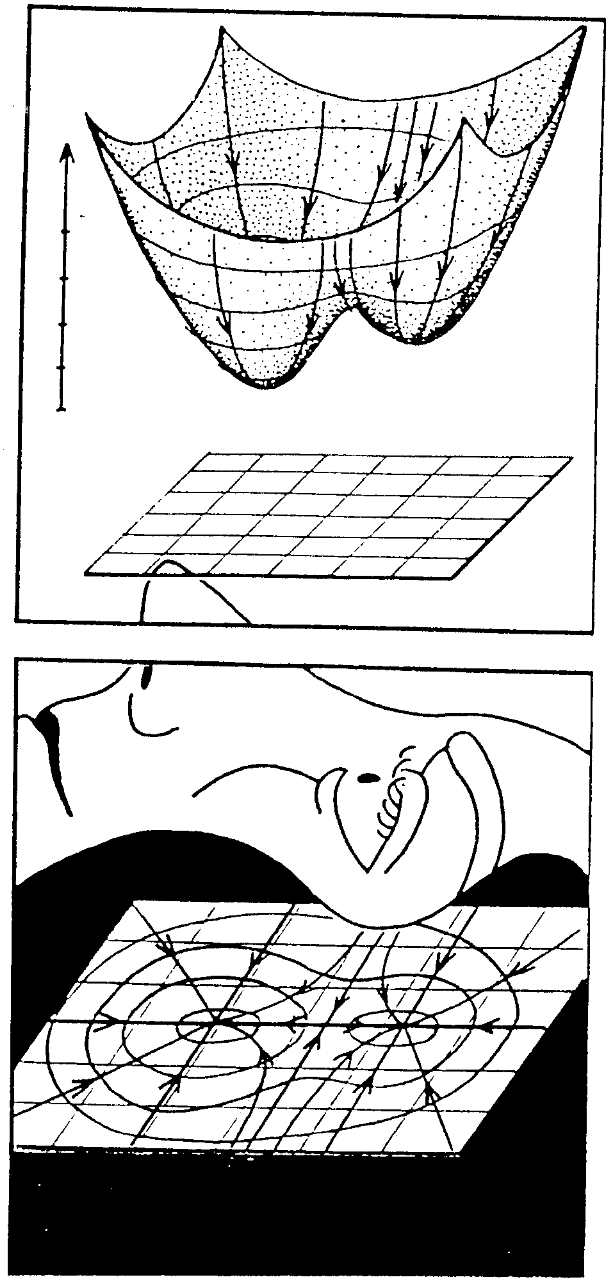


Figure 23

TRAJECTORIES

Place skiers on the upper edge of the snow bowls and let them schuss the fall line (take the steepest grades down) and note their tracks (trajectories). Their speed (vector of the vectorfield) is proportional to the steepness of the slope at each point.

Viewed by Ulla, the ski goddess, from far below, their tracks appear to move over the contour map at right angles to the contours.

These trajectories, perpendicular to the contours, comprise the phase portrait of the gradient dynamical system. Gradient systems, generally, are much like this example. Their limit sets are usually equilibrium points (fixed point attractors). A limit cycle is impossible, as the skier cannot keep going downhill and still return to some earlier point (except in an Escher print). Although gradient systems are useful in some elementary science problems, their usefulness is severely limited by the lack of cyclic and chaotic limit sets.

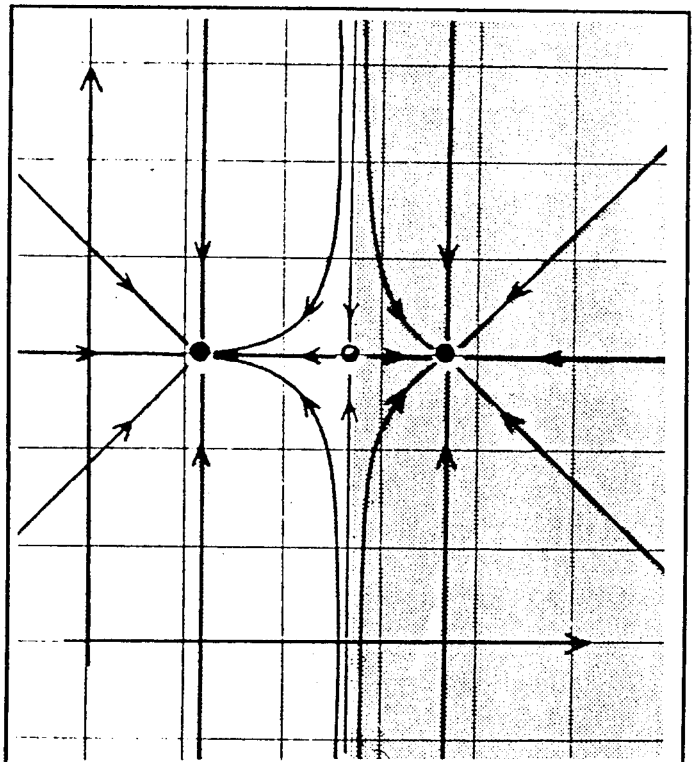


Figure 24. PHASE PORTRAIT

The phase portrait for the gradient system has two basins with a point attractor in each. Between them is a limit point of saddle type, a saddle point. This corresponds to the point on the ridge between the two valleys. The inset of the saddle point is the separatrix dividing the state space into the two basins.