The Relativity of Discovery: Hilbert's First Note on the Foundations of Physics*

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Hilbert's paper on "The Foundations of Physics (First Communication)," presented to the Göttingen Academy of Sciences on November 20, 1915,[1] is now primarily known for its parallel publication of essentially the same gravitational field equations of general relativity which Einstein published in a note on "The Field Equations of Gravitation," presented to the Prussian Academy of Sciences in Berlin five days later, on November 25, 1915.[2] An intense correspondence between Hilbert and Einstein in the crucial month of November 1915, furthermore, confronts the historian with a case of parallel research and with the associated problem of reconstructing the interaction between Hilbert and Einstein at that time.

Previous assessments of these issues have recently been challenged by Leo Corry, Jürgen Renn, and John Stachel who draw attention to a hitherto unnoticed first set of proofs for Hilbert's note.[3] These proofs bear a printer's stamp of December 6 and display substantial differences to the published version, in particular as regards the covariance of the theory and the discussion of the energy concept. They also do not yet contain the explicit form

*Arch. Hist. Exact Sci. 1998, in press.

[1][Hilbert 1915].

[2][Einstein 1915a].

[3][Corry, Renn and Stachel 1997].

of Einstein's gravitational field equations in terms of the Ricci tensor and its trace, the Riemann curvature scalar.

By focussing on the consequences of these findings for the reconstruction of Einstein's path towards general relativity — any possibility that Einstein took the clue for the final step towards his field equations from Hilbert's note is now definitely precluded — a number of questions about Hilbert's role in the episode, however, are left open. To what extent did Hilbert react to Einstein? What were Hilbert's research concerns in his note, and how did they come to overlap with Einstein's to some extent in the fall of 1915? How did Hilbert and Einstein regard each other and their concurrent activities at the time? What did Hilbert hope to achieve, and what, after all, did he achieve?

With these questions in mind I shall discuss in this paper Hilbert's first note on the "Foundations of Physics," its prehistory and characteristic features,[4] and, for heuristic purposes, I shall do so largely from Hilbert's perspective.

1 Hilbert and his research program

In contrast to Einstein's biography, Hilbert's life was curiously unaffected by the drastic historical changes associated with Germany's transition from the Kaiserreich to Weimar democracy to the Nazi regime, or, for that matter, by the first World War. Hilbert came to Göttingen in 1895, aged 33, as a young professor, and there he stayed for nearly fifty years. In Göttingen he lived almost all these years in the same house on Wilhelm-Weber-Straße 29, married to the same woman, and leading a scholarly life devoted to mathematical research and academic teaching. Hilbert never had a sabbatical, but he gave lecture courses, term after term, on various fields of mathematics, logic, and mathematical physics. These lectures are documented by more than a hundred *Vorlesungsausarbeitungen* kept in the Mathematics Institute

[4]For the sake of completeness, it should be noted that, in 1924, Hilbert published a revised version of his first note as the first part of a paper, titled also "Die Grundlagen der Physik" in the *Mathematische Annalen* ([Hilbert 1924]). The second part of this 1924 note was his second note on the "Foundations of Physics" from December 1916 ([Hilbert 1917]), revised in 1924 as well. A discussion of this 1924 republication of his first note is beyond the scope of the present paper. It would involve a discussion of Hilbert's subsequent work leading to the publication of his second note, of later investigations of the energy problem by Felix Klein and others, as well as of developments in unified field theory up to 1924.

of Göttingen University. In spring and fall, between terms, however, he used to leave Göttingen to spend vacations either on the Baltic or in the Swiss mountains.[5]

Intellectually, on the other hand, Hilbert's achievements have been called revolutionary, in particular as regards his work on the foundations of mathematics. And it would be wrong to assume that Hilbert stewed in his own juice in Göttingen. He did participate energetically in the scientific discussions of the day, and he also took active part in matters of scientific interest to the small but vital scientific community of Göttingen. It has often been said that the people of Göttingen in the era of Klein and Hilbert saw the town as one of the centers, if not as *the* center, of the scientific world.[6] Be that as it may, Hilbert's active intervention in matters of science policy aimed at having crucial people come to Göttingen. Leading scientists who worked creatively at the edge of current research in a field that was of interest to Göttingen mathematicians were invited to come there and inform Hilbert and his colleagues about the latest developments in the field.

A typical example is provided by Hilbert's first correspondence with Einstein.[7] In 1912 Hilbert was working on implications of his theory of linear integral equations for physics, in particular the kinetic theory of gases and radiation theory. This was a field where Einstein had published important contributions. So Hilbert sent Einstein a postcard asking for offprints of his papers. Some time later he sent Einstein a copy of his just published book on integral equations, and then invited him to come to Göttingen in the spring of the following year to attend a Wolfskehl symposium on the kinetic theory of matter. Einstein declined, saying he had nothing new to say and that he was also too busy.

Intellectually, Hilbert's science policy is matched by a characteristic feature of his perspective on the mathematical sciences captured by the catchword "the axiomatic method." The axiomatic method is characteristically reflective; it takes up whatever insight has been achieved into a field and tries to analyse, clarify and reformulate it. But it is not a vague, unspecific openness towards anything new in the sciences. In physics, too, Hilbert had rather firm judgments about important developments and the role he himself and his group might play in them.

[5]For biographical information on Hilbert, see [Blumenthal 1935] and [Reid 1970].

[6]For a discussion of the Göttingen mathematical community in the era of Klein and Hilbert, see [Rowe 1989].

[7]CPAE5, Docs. 378, 417.

Hilbert's first note on the Foundations of Physics from 1915 indeed represents the culmination of a lifelong interest and many years of active work in physics.[8] I would like to suggest that there are some characteristics of his understanding of physics which date to his early years and stay relatively invariant over the years.

1.1 Physics as a mathematical discipline

Hilbert's view of physics from a mathematician's perspective becomes quite explicit in remarks he made regarding the relationship between physics and geometry. Hilbert regarded geometry as a genuine branch of mathematics. But, originally, geometry was a natural science. Only it was no longer subject to experimental examination and had become mathematized, arithmetized and eventually axiomatized. For Hilbert, this development is not only an account of the factual historical development but also of the proper advancement of science, an advancement which should be furthered wherever possible.

Thus, as early as 1894, in a lecture on geometry which he gave while still in Königsberg, Hilbert wrote

> Geometry is a science which essentially has developed to such a state that all its facts may be derived by logical deduction from previous ones.[9]

And he immediately adds

> Completely different from, e.g. electricity theory or optics where even today new facts are still being discovered.[10]

Later in this lecture, in the course of discussing the axiomatic foundations of geometry, he presents the axiom of parallels and discusses the alternatives of Euclidean, hyperbolic and parabolic geometries. In this context he remarks

[8]For discussions of Hilbert's work in physics, see [Born 1922], [Corry 1996], [Corry 1997b].

[9] "Die Geometrie ist eine Wissenschaft, welche im Wesentlichen so weit fortgeschritten ist, dass alle ihre Thatsachen bereits durch logische Schlüsse aus früheren abgeleitet werden können." SUB Cod. Ms. Hilbert 541, p. 7.

[10] "Ganz anders wie z.B. die Electricitätstheorie oder Optik, in der noch heute immer neue Thatsachen entdeckt werden," ibid.

> Now also all other sciences are to be treated following the model of geometry, first of all mechanics, but then also optics and electricity theory.[11]

This is 1894. Very similar remarks are made in a lecture course on Euclidean geometry given in winter 1898/99. There Hilbert characterizes geometry as

> a natural science but of such a kind that its theory may be called a perfected one which, as it were, provides a model for the theoretical treatment of other natural sciences.[12]

In the same semester Hilbert also lectured on mechanics. This is in fact Hilbert's first lecture course dealing with physics proper.[13] In the introduction to this course, Hilbert again characterized geometry as a mathematical science which used to be a natural science. Regarding mechanics he then goes on:

> Also in mechanics the basic facts are accepted by all physicists. But the arrangement of the basic concepts nevertheless is subject to the changes in viewpoint. The structure is also far more complicated [than that of geometry]; even deciding what is simpler is something which depends on further discoveries. Hence, even today, mechanics cannot yet be called a purely mathematical discipline, at least not to the extent that geometry is.[14]

Given this state of affairs, Hilbert continues:

[11] "Nach dem Muster der Geometrie sind nun auch alle anderen Wissenschaften in erster Linie Mechanik, hernach aber auch Optik, Elektrizitätstheorie etc. zu behandeln," ibid., p. 92.

[12] " [...] eine Naturwissenschaft [...], aber eine solche, deren Theorie eine vollkommene zu nennen ist, die gleichsam ein Muster bildet für die theoretische Behandlung anderer Naturwissenschaften." SUB Cod. Ms. Hilbert 551, p. 1.

[13] Except for a lecture course on hydrodynamics held in Königsberg in summer 1887, cp. SUB Cod. Ms. Hilbert 522.

[14] "Auch in der Mechanik werden die Grundthatsachen von allen Physikern zwar anerkannt. Aber die Anordnung der Grundbegriffe ist dennoch dem Wechsel der Auffassungen unterworfen. Auch der Aufbau ist ein viel complicirterer und was das einfachere ist, zu entscheiden ist von weiteren Entdeckungen abhängig, so dass die Mechanik auch heute noch nicht, jedenfalls nicht in dem Masse wie die Geometrie als eine rein mathematische Disziplin zu bezeichnen ist." SUB Cod. Ms. Hilbert 553, p. 2.

> We must strive for it to become [a mathematical science]. We must extend the range of pure mathematics further and further, not only in our own mathematical interest but also for the sake of science as such.[15]

1.2 Physics and the axiomatic method

The lectures on Euclidean geometry and mechanics given in winter 1898/99 from which these quotes are taken immediately preceded the writing of Hilbert's famous and classic "Foundations of Geometry" published in June 1899 as part of a *Festschrift* on the occasion of the unveiling of the Gauss-Weber-monument in Göttingen.[16] Hilbert's axiomatic treatment of geometry as laid out in the *Festschrift* was immensely influential in the emergence of what is called the axiomatic method in mathematics, captured roughly by the postulates of independence, consistency, and completeness for systems of axioms, by the notion of implicit definition, and by the use of models in the logical analysis of axiomatic systems.

Hilbert's work in geometry was also influential for his own understanding of mathematics and physics. The quotes given above illustrate that Hilbert saw his axiomatic reformulation of Euclidean geometry as a model of the way in which physics was to be treated as well. For a proper understanding of Hilbert's subsequent work in physics some brief comments on the most explicit early programmatic formulation for this work are in order: the sixth of the 23 problems for future mathematical research which Hilbert formulated for the International Congress of Mathematicians in Paris in 1900, i.e., roughly a year after the *Festschrift.* This sixth problem explicitly reads

> 6. Mathematical Treatment of the Axioms of Physics. The investigations on the foundations of geometry suggest the problem: *To treat in the same manner, by means of axioms, those physical sciences in which already today mathematics plays an important part; in the first rank are the theory of probabilities and mechanics.*[17]

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[15]"Wir müssen streben, dass sie es wird. Wir müssen die Grenzen echter Mathematik immer weiter ziehen nicht nur in unserem mathematischen Interesse, sondern im Interesse der Wissenschaft überhaupt." ibid.

[16][Hilbert 1899]. The other part of this *Festschrift* was a discussion by Emil Wiechert of the "Foundations of Electrodynamics" ([Wiechert 1899]).

[17]"6. Mathematische Behandlung der Axiome der Physik. Durch die Untersuchungen

The formulation of the sixth problem might suggest that Hilbert rather specifically had in mind just the kind of logical analysis of basic assumptions, derived theorems, and their mutual interdependence which led to the axiomatic understanding of Euclidean geometry. His notes on mechanics show, indeed, that Hilbert focussed on such logical elements, for instance when he explicitly noted from the literature that not only do Kepler's laws follow from Newton's law, but that some converse assertion is also true. Namely, that if in the field of a central force all motions are conic sections or, alternatively, all bounded motions are closed, the force can only be the Newtonian $1/r^2$ or the harmonic force $\propto r$. Hence, to some extent, Newton's law conversely follows from Kepler's first law plus some additional assumptions.[18]

Another such example is taken from Boltzmann's mechanics from 1897.[19] There Boltzmann compares Gauss's dynamical principle of least action to the principle of virtual displacements by looking at the motion of a material point constrained to the convex side of a parabolic surface. While Gauss's principle allows the determination of the conditions when the point leaves the surface, the principle of virtual displacements fails to do so. In the latter case, additional assumptions have to be made.

These examples might suggest that Hilbert, in 1900, quite specifically had in mind a reflection on the received knowledge of classical point mechanics aiming at the identification of possible precise formulations of its dynamical principles and their derivable consequences. There is, however, another dimension to Hilbert's program for the axiomatic understanding of physics. Hilbert, indeed, gives some further explanation of what he had in mind. He mentions a number of "important investigations by physicists on the foundations of mechanics," namely the writings by Mach, Hertz, Boltzmann, and Volkmann.[20] Hilbert had studied Boltzmann's mechanics quite closely. He wrote:

> It is therefore very desirable that the discussion of the foundations

über die Grundlagen der Geometrie wird uns die Aufgabe nahegelegt, *nach diesem Vorbilde diejenigen physikalischen Disziplinen axiomatisch zu behandeln, in denen schon heute die Mathematik eine hervorragende Rolle spielt: dies sind in erster Linie die Wahrscheinlichkeitsrechnung und die Mechanik.*" [Hilbert 1900, p. 272], translation, slightly adapted, from [Hilbert 1901/02, p. 454].

[18]SUB Cod. Ms. Hilbert 553, pp. 27f.

[19]Ibid., p. 86. The example is taken from [Boltzmann 1897, pp. 223–225].

[20][Hilbert 1900, p. 272], [Hilbert 1901/02, p. 454]. The references are to [Mach 1889], [Hertz 1894], [Boltzmann 1897], and [Volkmann 1900].

> of mechanics be taken up by mathematicians also. Thus Boltzmann's work on the principles of mechanics suggests the problem of developing mathematically the limiting processes, there merely indicated, which lead from the atomistic view to the laws of motion of continua. Conversely one might try to derive the laws of the motion of rigid bodies by a limiting process from a system of axioms depending upon the idea of continuously varying conditions of a material filling all space continuously, these conditions being defined by parameters. For the question as to the equivalence of different systems of axioms is always of great theoretical interest.[21]

The specification of the vague term "mathematical treatment" or "axiomatic treatment" is given here by referring to the problem of a precise and rigorous mathematical formulation of the transition between discrete, atomistic conceptions and continuum mechanics. The important point is Hilbert's concern with a proper mathematical formulation of continuum mechanics.

In his early lectures, Hilbert always introduced a tripartite division of mechanics: the mechanics of a single mass point, the mechanics of systems of a finite number of mass points, and the mechanics of infinitely many mass points.[22] In Hilbert's understanding, the third part in itself was divided into continuum mechanics which did not only comprise elasticity theory and hydrodynamics but also the investigation of the motion of those fluids which "do not have all the properties of matter," such as electric and magnetic fluids, ether, and the motion of energy and entropy. The mechanics of infinitely many mass points, on the other hand, also included molecular physics, kinetic theory, the theory of ions and electrons as well as chemistry in general.[23]

[21]"[...]; es ist daher sehr wünschenswert, wenn auch von den Mathematikern die Erörterung der Grundlagen der Mechanik aufgenommen würde. So regt uns beispielsweise das Boltzmannsche Buch über die Prinzipe der Mechanik an, die dort angedeuteten Grenzprocesse, die von der atomistischen Auffassung zu den Gesetzen über die Bewegung der Continua führen, streng mathematisch zu begründen und durchzuführen. Umgekehrt könnte man die Gesetze über die Bewegung starrer Körper durch Grenzprozesse aus einem System von Axiomen abzuleiten suchen, die auf der Vorstellung von stetig veränderlichen, durch Parameter zu definierenden Zuständen eines den ganzen Raum stetig erfüllenden Stoffes beruhen — ist doch die Frage nach der Gleichberechtigung verschiedener Axiomensysteme stets von hohem prinzipiellen Interesse." ibid.

[22]See, e.g., SUB Cod. Ms. Hilbert 553, p. 7.

[23]See Hilbert's first lecture course on Continuum Mechanics from winter 1902/03 and

A decision on the question whether matter ultimately consists of atoms or whether it is some continuously extended substance is avoided. One of the major sources of Hilbert's knowledge of mechanics, however, Kirchhoff's textbook from 1876, deals extensively with a continuum mechanics under the explicit assumption that matter fills space continuously, "as it appears to do."[24]

The mechanics of infinitely many mass points (be it in its continuum version or in its discrete, atomistic version), in any case, comprises all of physics. The sixth problem, therefore, in some sense simply formulates the task of mathematizing physics and of making it just one more branch of mathematics.

Put this way, the Paris problem may seem preposterously ambitious. Not necessarily so for Hilbert. He had solved some longstanding mathematical problems by bringing together hitherto unrelated branches of mathematics. His proof of the finite basis theorem had put an abrupt end to decades of active mathematical research in invariant theory by solving a central problem on a higher level of abstraction, and a similar breakthrough was achieved by his monumental *Zahlbericht* which systematized and summarized nineteenth-century knowledge in number theory.[25] And, in the concluding passage of his Paris lecture, Hilbert explicitly expressed a firm belief in the unity of the mathematical sciences.

> Mathematical science is in my opinion an indivisible whole, an organism whose vitality is conditioned upon the connection of its parts. For with all the variety of mathematical knowledge, we are still clearly conscious of the similarity of the logical devices, the relationship of the ideas in mathematics as a whole and the numerous analogies in its different departments.[26]

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summer 1903 as documented by an *Ausarbeitung* by Berkowski kept at the Library of the Mathematics Institute of Göttingen University, p. 2; see also SUB Cod. Ms. Hilbert 553, pp. 145–147.

[24]"wie sie es zu thun scheint" [Kirchhoff 1877, p. III]. Hilbert's excerpts from Kirchhoff are kept in SUB Cod. Ms. Hilbert 516.

[25]See [Blumenthal 1935] and [Weyl 1944] for a discussion of Hilbert's mathematical achievements.

[26]"[...]; die mathematische Wissenschaft ist meiner Ansicht nach ein unteilbares Ganze, ein Organismus, dessen Lebensfähigkeit durch den Zusammenhang seiner Teile bedingt wird. Denn bei aller Verschiedenheit des mathematischen Wissensstoffes im einzelnen gewahren wir doch sehr deutlich die Gleichheit der logischen Hilfsmittel, die Ver-

Regarding the advancement of mathematical physics, Hilbert believed that the mathematician's task was the deliberate application of advanced and sophisticated mathematical methods and concepts. In his Paris lecture, he mentioned Lie's theory of infinite transformation groups as a possible means for systematically distinguishing systems of axioms for physics.[27] He also believed that the calculus of variations was indeed just such a powerful, "logical device" and that it could be utilized for a mathematically precise formulation of mechanics, in particular for continuous systems.[28] Another field of mathematical expertise relevant for the advancement of mathematical physics was, of course, the theory of both ordinary and partial differential equations.

So much for Hilbert's programmatic formulation of 20th century mathematical physics. He was 38 and president of the *Deutsche Mathematikervereinigung* when he delivered his address at the International Congress of some two hundred mathematicians from all over the world, an address which would establish his role as the "Generaldirector" of mathematical science, as Minkowski put it in an enthusiastic letter to Hilbert.[29] (Einstein, at the time of Hilbert's Paris speech, was 21 years of age and was just spending a few days of vacation with his mother and sister in the Swiss resort village of Melchtal after having passed the oral examinations for his ETH Diplom as "Fachlehrer in mathematischer Richtung."[30])

2 Prehistory

Hilbert, at that time, was a well-known if not yet world-famous mathematician. In his five years as professor in Göttingen Hilbert had already had a first occasion to decline a call to the university of Leipzig. Another such occasion soon followed when Hilbert was offered the prestigious chair of Lazarus Fuchs at Berlin University.[31] Again he declined, but he succeeded in convincing the Prussian ministry to create an extra professorship for mathematics

wandtschaft der Ideenbildungen in der ganzen Mathematik und die zahlreichen Analogien in ihren verschiedenen Wissensgebieten." [Hilbert 1900, p. 297], [Hilbert 1901/02, p. 478].

[27][Hilbert 1900, p. 273], [Hilbert 1901/02, p. 454].

[28]For a discussion of Hilbert's work in the calculus of variations, see [Thiele 1997]; see also [Blum 1994].

[29][Minkowski 1973, p. 130].

[30]CPAE1, Docs. 68–71.

[31][Blumenthal 1935, pp. 401, 406f].

which would allow him to get Minkowski to come to Göttingen. His plan worked and Minkowski came in 1902; and this is also the time, biographically, when Hilbert again started to work on physics. He begins with a two-semester course on continuum mechanics. In the following semester Hilbert and Minkowski also gave a joint seminar on the general problem of the mechanical stability of continuous systems. In this context, Hilbert also started to do research, as witnessed by a talk to the Naturforscherversammlung in Kassel in 1903. The talk was "On Mechanics of Continua" and dealt with the problem of the mechanical stability of a liquid in a vessel.[32]

2.1 The period 1902–1915

Notwithstanding Hilbert's early research work on continuum mechanics, and notwithstanding a series of lecture courses and joint seminars on the mathematical problems of mechanics, electromagnetism and other fields of physics, Göttingen mathematical physics, as far as Hilbert was concerned, was a task pursued predominantly by Minkowski. Hilbert in these years was working on his theory of integral equations. After all, the axiomatization of physics was just one problem among 22 others on his list.

At that time, Göttingen was a stronghold of the so-called electromagnetic world view.[33] In 1902 and 1903, the Göttingen theoretician Max Abraham had published programmatic papers on the dynamics of the electron.[34] He had shown that the inertial mass of the electron may completely be accounted for on the basis of Maxwellian electromagnetism.

Assuming that the electron is a small rigid sphere with uniform charge distribution, Abraham had computed the work to be done in changing the energy and momentum of the electron's self-field when it is accelerated by an external field. Given that the electron was regarded as the ultimate constituent of matter Abraham's result opened up the perspective of a complete reinterpretation of the concepts of Newtonian mechanics in terms of

[32]SUB Cod. Ms. Hilbert 593. The problem of the talk was how to control mathematically the infinitely many degrees of freedom if you investigate the stability of the liquid by looking at small deviations of its center of mass from its equilibrium position given that neither the shape of the surface nor its boundary conditions are fixed.

[33]See [Jungnickel and McCormach 1986], pp. 227–245, for a discussion of the electromagnetic worldview in the setting of Germany's theoretical physics at the turn of the century.

[34][Abraham 1902a], [Abraham 1902b], [Abraham 1903]; for a historical discussion of Abraham's electron theory, see [Goldberg 1977].

Maxwellian electrodynamics and the foundation of physics on the basis of Maxwell's electromagnetic field theory. Moreover, in doing the calculations, Abraham had obtained an expression for the inertial mass of the electron which turned out to depend on the electron's velocity. And these results seemed to be in good agreement with experimental data obtained by the Göttingen experimental physicist Kaufmann on beta and cathode rays.[35] This confirmation of Abraham's theory was "certainly one of the most important results of modern physics," as Lorentz observed in 1909.[36]

Minkowski's subsequent role in the discussions of current theoretical problems in electrodynamics and his work on special relativity and four-dimensional electrodynamics[37] may have helped to effect a reorientation of Hilbert's global perspective on physics. Hilbert, too, gradually changed from his predominant concern with a mathematically sound formulation of continuum mechanics to an electromagnetic world view, which assigned some crucial importance to the theoretical understanding of the electron.[38] After Minkowski's death and Born's work on the rigid body problem in special relativity,[39] it was two of Hilbert's students, Erich Hecke and Wilhelm Behrens, who continued to work out details of special-relativistic electron dynamics on the basis of the Abraham-Born theory of the rigid spherical electron.[40]

During this time, however, Hilbert himself worked predominantly on mathematical problems, a major one being his theory of linear integral equations. A decisive turn in Hilbert's research interests occurred in 1912, when he had just finished a major monograph on this subject, summarizing the results and fruits of his research of the previous eight years in a systematic exposition.[41] Working on the final chapters of his book, he considered applications of his theory for physics and discovered that it may provide a means of clarifying the foundations of kinetic gas theory, and shortly later found that it also could be applied to radiation theory. These insights called his old dreams of mathematizing physics to life again. From now on, Hilbert devoted his time during the next few years almost exclusively to mathemat-

[35][Kaufmann 1902].

[36][Lorentz 1909, p. 43].

[37]For recent historical discussions of Minkowski's work in relativity, see [Corry 1997a] and [Walter 1998].

[38]Cp. the discussion in [Corry 1998].

[39][Born 1909a], [Born 1909b] [Born 1909c], [Born 1910].

[40][Behrens and Hecke 1912].

[41][Hilbert 1912].

ical physics and after more than three years, in the fall of November 1915 his efforts promised to result in a major breakthrough, perhaps comparable to his achievements in invariant theory or geometry: he realized that a novel electromagnetic theory of the electron put forward in 1912 by the Greifswald physicist Gustav Mie could be incorporated into the same mathematical framework as another completely novel theory of the gravitational field published just recently by the young and rather creative theoretician Albert Einstein. This would give Hilbert the chance of putting forward a unified mathematical description of both the gravitational and electromagnetic fields and, at the same time, a framework to compute the dynamics of the electron and possibly to explain the quantum features of the Bohr atom.

Hilbert's and Klein's science policy of systematically acquainting themselves and their fellow Göttingen mathematicians with the most recent research had paid off handsomely. Mie's theory was published in three installments in 1912.[42] But even before the last communication was issued, Max Born gave a report on this theory to the Göttingen mathematical society.[43] Born himself continued to work on Mie's theory, reinterpreting it in terms of Minkowski's four-dimensional formulation and working out formal relations to special-relativistic elasticity theory. He gave another talk on Mie's theory and on some of his own work a year later in late 1913.[44] At that time, a report on Einstein's and Grossmann's just recently published *Entwurf einer verallgemeinerten Relativitätstheorie und einer Theorie der Gravitation* was given as well.[45] Einstein's theory with its strange mathematical intricacies must have induced Hilbert to try again to invite Einstein to Göttingen to learn about this theory at first hand. In summer 1915 he succeeded, and in the first week of July Einstein gave a course of six Wolfskehl lectures on his

[42][Mie 1912a], [Mie 1912b], [Mie 1913]. For discussions of Mie's theory, see [Pauli 1921, pp. 754–759], [Vizgin 1994, pp. 26–38].

[43]Born's report "on the works by Mie on the theory of matter" was given on December 17 (JDM 22 (1913), p. 27), [Mie 1913] was issued on December 31, 1912.

[44]Born gave a report on "Mie's theory of matter" to the Göttingen Mathematical Society on November 25, and lectured on some of his own work, "carried out in connection with Mie's theory of matter," on December 16 (JDM 22 (1913), p. 207). [Born 1914] was presented to the Göttingen Academy by Hilbert on December 20, 1913. Hilbert had also asked for offprints of Mie's papers in the fall of 1913, see Mie to Hilbert, 22 October 1913, SUB Cod. Ms. Hilbert 254/1.

[45][Einstein and Grossmann 1913] was published in June 1913 (CPAE4, p. 340); a report on "the work by Einstein and Grossmann on gravitation theory" was given to the Göttingen Mathematical Society on December 9, 1913 (JDM 22 (1913), p. 207).

general theory of relativity.[46] Hilbert and Einstein were enthusiastic about each other on this their first encounter.[47] At the end of the summer term, Hilbert was left with a number of rather interesting but intricate papers on mathematical physics which called for competent mathematical analysis. He would soon find not only exciting analogies and connections between those two theories but also pinpoint flaws in Einstein's rather pedestrian way of dealing with the mathematics of his gravitation theory.

2.2 Mid-July to Mid-November 1915

Unfortunately, very little is known about Hilbert's whereabouts and intellectual preoccupations in the late summer and fall of 1915 after Einstein's visit. One of the last documents from this summer is a letter to Schwarzschild, dated 17 July. In this letter Hilbert wrote

> We had a lot of scientific business here during the two war semesters. Almost all mathematics and physics lecturers incl. Voigt and Tammann participate in my seminar; the main instigator is, of course, Debye. During the summer we had as guests one after the other: Sommerfeld, Born, Einstein. In particular, the lectures of the latter on gravitation theory were memorable.[48]

Four days later, a meeting in preparation for Emmy Noether's habilitation was held in Göttingen, and it is known that Hilbert, the most fervent propo-

[46] GM 1916, p. 13. Leo Corry first found notes on a part of Einstein's lectures taken by an unknown auditor and preserved in the Hilbert archives (SUB Cod. Ms. Hilbert 724). These notes have meanwhile been published as Appendix B to CPAE6, pp. 586–590. On June 29, 1915, Einstein lectured "on gravitation" at the Göttingen Mathematical Society (JDM 24 (1915), p. 68).

[47] For Einstein's reaction to Hilbert, see Einstein to Zangger, 7 July 1915, CPAE8, Doc. 94; Einstein to Sommerfeld, July 15, 1915 ([Hermann 1968, p. 30]) CPAE8, Doc. 96; Einstein to W. and G. de Haas, August 1915 ([Pais 1982, p. 259]) CPAE8, Doc. 110. For Hilbert's reaction, see the quotes given below.

[48] "Wir haben hier während der beiden Kriegssemester vielen wissenschaftlichen Betrieb gehabt. An meinem Seminar nehmen fast alle math. u. phys. Dozenten incl. Voigt u. Tammann teil; der Hauptmacher ist natürlich Debye. Während des Sommers hatten wir hier zu Gast der Reihe nach: Sommerfeld, Born, Einstein. Besonders die Vorträge des letzteren über Gravitationsth. waren ein Ereigniss." Hilbert to Schwarzschild, July 17, 1915, SUB Cod. Ms. Schwarzschild 331/7.

nent and advocate of Noether's habilitation, attended.[49] Hilbert's presence was also needed since the affair was more than routine academic business. It would set a precedent and implied breaking a decree issued in 1908 by the Prussian ministry against the habilitation of women at Prussian universities.[50] After this July meeting it is not known what Hilbert did or where he was.

Then there is a postcard from Hilbert's student Richard Bär who later worked out Hilbert's relativity lectures, dated August 17, 1915.[51] Bär had visited Hilbert in Switzerland and the two sent greetings to Hilbert's wife who had remained in Göttingen.

Six years later, Felix Klein reports in a letter to Pauli that Hilbert had the decisive insight ("die entscheidende Gedankenwendung") in the fall of 1915 in Rügen, and adds that Sommerfeld should know more about this.[52] Klein's recollection is corroborated by a letter from Erich Hecke to Hilbert, dated 16 October, in which he thanks him for a postcard "von Rügen aus."[53] While it is thus probable that Hilbert did spend some time during the fall of 1915 on Rügen, we don't know exactly when.[54]

The next thing we know about Hilbert is that he spent a week in Munich in mid-October. On his way back, he met Schwarzschild's mother in the train, and took the occasion to write a postcard to Schwarzschild, dated Oct. 23:

> Dear S[chwarzschild].
> Just as we were coming back from an eight-day visit to Munich we met your mother in the train.—During the summer semester we had many scientific guests in Göttingen, among them also

[49][Tollmien 1991, p. 15]. Tollmien gives a detailed historical account of Emmy Noether's habilitation with extensive quotes of the extant archival documents.

[50]As it turned out this first attempt eventually failed. Emmy Noether's habilitation was only achieved in Göttingen in 1919 after the end of the war and after the German November revolution, see [Tollmien 1991].

[51]SUB Cod. Ms. Hilbert 772/1.

[52]Klein to Pauli, 8 Mai 1921, [Pauli 1979, p. 31].

[53]SUB Cod. Ms. Hilbert 141/5.

[54]The question may be relevant because Einstein also spent a couple of weeks in Rügen in the second half of July, i.e., after he gave his talks in Göttingen. There is hence a slight chance that Hilbert was in Rügen at the same time as Einstein and that the two continued their discussion there. However, as far as I know there is no further indication in the correspondence that would corroborate this surmise.

> Einstein, [which] was highly interesting. The "astronomical" Freundlich was with us as well. The astronomers, I think, must now leave everything else aside and only try to verify or refute Einstein's gravitation law!
> Yours sincerely.[55]

With his return to Göttingen, a rather busy time began for Hilbert with the daily routine of the winter term. A first Academy meeting which always took place on Saturdays was held on October 23.[56] Then he had to give his lectures: Mondays he lectured for two hours on differential equations and gave a two-hour seminar on the structure of matter in the afternoon together with Debye.[57] On Monday evenings he attended the physics colloquium.[58] Tuesdays meetings of the Göttingen Mathematical Society were held.[59] On Thursday mornings another two hours of lectures on differential equations were scheduled, and in the afternoon there were the weekly hikes of Hilbert, Klein and Runge where science policy matters had to be discussed.[60]

On the following Friday, October 29, a commission meeting concerning the habilitation of Emmy Noether took place. Another meeting of the Mathematics-Physics Department was scheduled for the following week, on Saturday, November 6. On Tuesday, November 9, Emmy Noether gave a talk on transcendental numbers to the Göttingen mathematical society.[61] On the day after that, the Historical-Philosophical Department discussed Noether's

[55] "Lieber S[chwarzschild]. Eben von einem 8tägigen Ausflug von München kommend, treffen wir Ihre Frau Mutter im Zuge.—Im Sommersemester haben wir in Göttingen viele wissenschaftliche Gäste gehabt, darunter auch Einstein, war hochinteressant. Auch der astronomische Freundlich war bei uns. Die Astronomen, meine ich, müssen nun Alles liegen lassen u. nur darnach trachten das Einsteinsche Gravitationsgesetz zu bestätigen oder widerlegen! Haben Sie herzlichste Grüße." Hilbert to Schwarzschild, October 23, 1915, SUB Cod. Ms. Schwarzschild 331/6; see also Pringsheim to Hilbert, 25 October 1915, SUB Cod. Ms. Hilbert 318/6.

[56] GAA Chron 2.1, Vol. 6, 410.

[57] [Verzeichnis 1915, pp. 14, 17]. An *Ausarbeitung* of his lecture course on differential equations is kept at the Mathematics Institute of Göttingen University.

[58] See Hilbert to Einstein, 13 November 1915, CPAE8, Doc. 140.

[59] The meetings started on November 2 with a session on "Ferienberichte," JDM 24 (1915), p. 111.

[60] See, e.g., [Hilbert 1909, p. 101], [Blumenthal 1935, p. 407], [Runge 1949, p. 123].

[61] JDM 24 (1915), p. 111, [Dick 1970, p. 14]. Dick quotes a letter by Emmy Noether to Fischer: "Das hat sich sogar der hiesige Geograph angehört, für den es ein bißchen sehr abstrakt war; die Fakultät will sich in ihrer Sitzung von den Mathematikern keine Katze im Sack verkaufen lassen." ibid.

petition for habilitation; these colleagues were particularly conservative.[62]

Despite all these matters to be seen of, Hilbert still tried to work out the implications of his "entscheidende Gedankenwendung," as Klein had called it, implying generally covariant field equations for the gravitational and for the electromagnetic fields, some mathematical relations between these field equations, a discussion of the energy theorem in this framework, and the prospect of finding a solution to the non-linear generalized Maxwell equations that would correctly describe the electron. However, on the previous weekend Hilbert had received an alarming letter from Einstein, who had sent him the proofs of his first November communication,[63] which introduced Einstein's return to general covariance[64] and also presented a modified set of gravitational field equations that, however, were not yet generally covariant. Einstein wrote:

> With the same mail I am sending you the proofs of a paper in which I modified the gravitation equations after I myself had realized some four weeks ago that my former proof was flawed. Colleague Sommerfeld wrote to me that you also found a hair in my soup which made it unpalatable to you. I am curious whether you will come to like this new solution.[65]

Hilbert responded with a friendly letter, now lost. But matters got worse on the following weekend when he received a second letter from Einstein[66] in which Einstein reported further progress on achieving general covariance in his theory of gravitation. In his second communication to the Berlin Academy,[67] Einstein now had published generally covariant field equations using the Ricci tensor but at the price of assuming that the trace of the

[62][Tollmien 1991], cp. note 49.

[63][Einstein 1915b].

[64]There is an extensive literature on Einstein's path towards General Relativity, see, in particular the classical accounts of [Stachel 1980] and [Norton 1984]. See also [Renn and Sauer 1998] and the references cited therein.

[65]"Mit gleicher Post sende ich Ihnen die Korrektur einer Arbeit, in der ich die Gravitationsgleichungen abgeändert habe, nachdem ich selbst vor etwa 4 Wochen erkannt hatte, dass mein bisheriges Beweisverfahren ein trügerisches war. Kollege Sommerfeld schrieb mir, dass auch Sie in meiner Suppe ein Haar gefunden haben, das sie Ihnen vollkommen verleidete. Ich bin neugierig, ob Sie sich mit dieser neuen Lösung befreunden werden." Einstein to Hilbert, 7 November 1915, CPAE8, Doc. 136.

[66]Einstein to Hilbert, 12 November 1915, CPAE8, Doc. 139.

[67][Einstein 1915c].

energy-momentum tensor vanish as it does for the electromagnetic energy-momentum tensor. In his letter Einstein alluded to this consequence by pointing out that, by this hypothesis, gravitation must play a fundamental role in the constitution of matter.

But this touched on what Hilbert saw as his own original insight. Alarmed, Hilbert decided to take action. He announced a lecture on the "fundamental equations of physics" in the Göttingen Mathematical Society where he wanted to present his own recent investigations.[68] He called Emmy Noether for assistance. She wrote to Fischer in Erlangen

> Invariant theory here is now trump; even [Paul] Hertz, the physicist, studies Gordan-Kerschensteiner; Hilbert wants to talk next week about his Einsteinian differential invariants, and so the Göttingers must get up to speed.[69]

And on Saturday, November 13, he sent two postcards to Einstein asking him to come to his lecture on the following Tuesday, November 16, or better still to the physics colloquium on Monday evening, inviting him to stay at Hilbert's house.[70] Einstein declined.[71] He was, in fact, just then working out what proved to be perhaps his greatest breakthrough, the correct computation of the perihelion advance of Mercury on the basis of his new field equations.

So Hilbert had to report his findings in correspondence to Einstein, unfortunately lost. He probably sent Einstein the manuscript of his lecture to the Göttingen Mathematical Society, or a summary of its main points.[72] But he realized that he did not have any more time to carefully work out the

[68] JDM 24 (1915), p. 111: "Dritte Sitzung am 16. November. Hilbert, Grundgleichungen der Physik."

[69] "Invariantentheorie ist jetzt hier Trumpf; sogar der Physiker [Paul] Hertz studiert Gordan-Kerschensteiner; Hilbert will nächste Woche über seine Einsteinschen Differentialinvarianten vortragen, und da müssen die Göttinger doch etwas können." [Dick 1970, p. 14]. Unfortunately, Dick does not give a source for this letter and dates it only unspecifically to November 1915. The textbook on invariant theory referred to is [Kerschensteiner 1885], [Kerschensteiner 1887]. In summer 1915 Hertz corresponded with Einstein about the latter's hole argument; see [Howard and Norton 1993], for a discussion of this correspondence.

[70] Hilbert to Einstein, 13 November 1915, CPAE8, Doc. 140.

[71] Einstein to Hilbert, 15 November 1915, CPAE8, Doc. 144.

[72] In his letter of November 18, Einstein responded to this lost piece of correspondence, that, as far as Einstein could tell, Hilbert's system was equivalent to the one he, Einstein, had found in the preceding weeks and presented to the Berlin Academy (Einstein to Hilbert, 18 November 1915, CPAE8, Doc. 148).

consequences of his own theory as he had hoped to do, in particular that he would not succeed in obtaining a solution to the electron problem in time. The next occasion for presenting a communication to the Göttingen Academy was Saturday, November 20, but Hilbert had already sent his manuscript on the "fundamental equations of physics" to the printer a day ahead, on November 19, a rather unusual irregularity in the Academy's proceedings.[73] And he must have been glad he did, since on that day he received notice from Einstein that he had succeeded in calculating the correct perihelion advance.[74] To that Hilbert could only respond with a "herzliche Gratulation," and the remark

> If I could do the calculations as rapidly as you, the electron would have to surrender and the hydrogen atom would have to produce a letter from home excusing it from not radiating.[75]

3 The proofs and the published paper

Hilbert's note was published under the title "The Foundations of Physics (First Communication)" in the last issue of the 1915 volume of the *Nachrichten*

[73] The invitation for the meeting of 20 November was issued on November 15 and was, as always, circulated among the members to confirm their participation and announce any communications they intended to present at the meeting. Into this invitation, Hilbert wrote: "Hilbert legt vor in die Nachrichten: Grundgleichungen der Physik." (GAA Chron 2.1, Vol. 6, 411). For each communication to the "Nachrichten", the "Journal für die 'Nachrichten' der Gesellschaft der Wissenschaften in Göttingen," (GAA Scient 66, Nr. 2), lists author and title of the paper, the member of the Academy who presented it for publication, and the date of presentation. In addition, there are columns "To the printer" ("Zum Druck"), "correction" ("Korrektur"), and an untitled column which has the information about the issue in which the article was finally published. For Hilbert's communication, entry Nr. 730 of the "Journal," the title "Die Grundgleichungen der Physik (Erste Mitteilung)" was later corrected to "Die Grundlagen der Physik (Erste Mitteilung)." In the period covered by the "Journal", 1912–1935, there were eight items out of several hundred which were given to the printer *before* being presented to the Academy.

[74] Einstein to Hilbert, 18 November 1915, CPAE8, Doc. 148. On that day Einstein had presented his third communication ([Einstein 1915d]) to the Berlin Academy. See [Earman and Janssen 1993] and CPAE4, pp. 344–359, for a discussion of Einstein's paper on the perihelion advance of Mercury and for a reconstruction of the rapidity of his calculations.

[75] "Wenn ich so rasch rechnen könnte, wie Sie, müsste bei meinen Gleichg das Elektron kapituliren und zugleich das Wasserstoffatom sein Entschuldigungszettel aufzeigen, warum es nicht strahlt." Hilbert to Einstein, 19 November 1915, CPAE8, Doc. 149.

of the Göttingen Academy of Sciences.[76] As mentioned above, attention has recently been drawn to a first set of proofs for this note.[77] This first set bears a printer's stamp of December 6 and displays substantial differences from the published version, in particular with regard to the discussion of the concept of energy and the covariance of the theory. In contrast to standard accounts of the history of General Relativity it was pointed out that, in this first version, Hilbert still believed in Einstein's argument for the necessity of distinguishing between what Hilbert called world parameters and space-time coordinates, i.e., the necessity of introducing the so-called adapted coordinate systems of Einstein's *Entwurf* theory, which restrict the general covariance of the theory. Corry, Renn, and Stachel, in particular, emphasize the fact that the proofs do not yet contain the explicit form of the gravitational field equations in terms of the Ricci tensor and the Riemann curvature scalar.

Despite these important differences, however, there are also a number of characteristic features of Hilbert's theory as presented in the published version that are already found in the proofs. For a reconstruction of Hilbert's

[76]"Die Grundlagen der Physik (Erste Mitteilung)" [Hilbert 1915]. The published version was issued on 31 March 1916 ([Corry, Renn and Stachel 1997, p. 1271]) but apparently Hilbert had received offprints or proofs of the final version by mid-February 1916 which he sent to various colleagues, see, e.g., P. Hertz to Hilbert, 17 February 1916, SUB Cod. Ms. Hilbert 150/2, L. Königsberger to Hilbert, 20 February 1916, SUB Cod. Ms. Hilbert 187/13, G. Mie to Hilbert, 13 February 1916, SUB Cod. Ms. Hilbert 254/2. These offprints, however, apparently did not yet bear the pagination of the published version but had page numbers 1ff., see Hilbert to Schwarzschild, 26 February 1916, SUB Cod. Ms. Schwarzschild 331/8. [Klein 1917] als refers to page numbers 1ff. The page references in Klein's note were updated to the page numbers 395ff. in the reprint of his Collected Papers. [Einstein 1916a, p. 810] referred to a p. 3 of Hilbert's note.

[77][Corry, Renn and Stachel 1997]. The proofs are preserved in SUB Cod. Ms. Hilbert 634 (in the following referred to as "proofs"). They consist of 13 pages of consecutive pagination. Pages are printed on both sides. Pp. 1/2, [7/8], and 13 are on a single sheets, pp. 3/4 and 5/6 as well as pp. 9/10 and 11/12 are on folded signature sheets. From the single sheet which contains pp. [7/8] a piece was cut off from the top such that approximately 10 lines are missing on the top of these pages including the page numbers. Pp. 1, 9, and 13 display a printer's stamp by the "Dieterichsche Univ.-Buchdruckerei W. Fr. Kaestner Göttingen" with the date of December 6, 1915. On the top of p. 1 Hilbert added in dark ink "Erste Korrektur meiner ersten Note." Underneath these words, some words had been written which were later erased. The pages contain some correction marks but also some marks in pencil which were later erased as well. On the top right corners of pp. 1, 3, and 7, Roman numbers I, II, III were added in ink. It is apparently these three sheets which Hilbert sent to Klein in March 1918, see Hilbert to Klein, March 7, 1918 ([Hilbert and Klein 1985, p. 144]).

part in this episode, it seems worthwhile to point out these characteristics here as well.

For one, Hilbert did not introduce any changes in the concluding remarks of his paper, which distinctly echo his programmatic concern of the 1900 Paris problems. Summarizing the main points of his note, Hilbert concluded that

> the possibility draws near that physics will become, in principle, a science of the same kind as geometry: certainly the most magnificent glory of the axiomatic method, which here, as we see, takes into its service the most powerful devices of analysis, i.e., the calculus of variations and invariant theory.[78]

The emphasis of this concluding passage suggests that Hilbert saw his communication as a culmination of his work in physics along his program of an axiomatic treatment of physics. Using advanced and sophisticated mathematical techniques, his aim was to promote theoretical physics to that mathematized and axiomatized state that geometry had already reached. Hilbert's note, as he claimed, represented a major achievement in this endeavour.

3.1 The gravitational field equations

Hilbert's emphasis on the role of invariant theory and the calculus of variations points to a feature of his theory which distinguishes it from Einstein's *Entwurf* theory and which is also already found in the proofs. In the *Entwurf* theory, as expounded in [Einstein 1914], Einstein gave a concise and independent introduction to the mathematics of the "absolute differential calculus" of Ricci and Levi-Civita in a special section headed "From the Theory of Covariants."[79] But while the concepts of this section do provide a framework for the formulation of a theory which is generally covariant, i.e., covariant with respect to arbitrary coordinate transformations, gravitational field equations which manifestly are not generally covariant were derived from a variational

[78]"[...] die Möglichkeit naherückt, daß aus der Physik im Prinzip eine Wissenschaft von der Art der Geometrie werde: gewiß der herrlichste Ruhm der axiomatischen Methode, die hier wie wir sehen die mächtigsten Instrumente der Analysis, nämlich die Variationsrechnung und Invariantentheorie, in ihre Dienste nimmt." Proofs, p. 13, [Hilbert 1915, p. 407].

[79]"Aus der Theorie der Kovarianten." [Einstein 1914, pp. 1034–1054]. For a discussion of the historical background and context of Einstein's mathematics, see [Reich 1994].

principle in section D of that paper by introducing a restriction of the theory to so-called adapted coordinate systems. In his first November communication Einstein had then replaced these field equations by a different set of field equations, also derived from a variational principle but still not generally covariant. The field equations of his second November communication, equating the Ricci tensor to the energy-momentum tensor, then finally were generally covariant but these field equations were not derived from a variational principle.

Hilbert, in his communication, introduced gravitational field equations which are derived from a variational principle and which are generally covariant. Thus, in contrast to Einstein's *Entwurf* theory and in contrast to Einstein's first November communication, he did not write down gravitational field equations of restricted covariance, and, in contrast to Einstein's second November communication, Hilbert did formulate the generally covariant field equations in terms of a variational principle.

In fact, Hilbert based his theory on two "axioms." The first axiom introduces an action integral

$$\int H\sqrt{g}d\tau, \tag{1}$$

where g is the determinant of the metric and $d\tau = dw_1dw_2dw_3dw_4$ with "world parameters" w_i uniquely determining the "worldpoints," i.e., the coordinates of a four-dimensional manifold.[80] The world function H is postulated to be a function depending on the components of the metric tensor, its first and second derivatives, as well as on the components of the electromagnetic four-potential and its first derivatives. The axiom specifically postulates that the laws of physics are determined by the condition that the variation of the integral (1) vanish. The second axiom then postulates that H transforms as an invariant under arbitrary transformations of the coordinates w_i.[81]

In the proofs, the field equations are not explicitly specified. They are only given as Lagrangian derivatives of the undetermined but invariant "world

[80]In the discussion of Hilbert's note, I will closely follow Hilbert's own notation. In particular, coordinates will be written with a subscript index, no summation convention is implied, and coordinate derivatives are indicated by simple subscript indices. A modern reader should also be aware of the fact that, although Hilbert does distinguish between contravariant, superscript and covariant, subscript indices (except for the coordinates), he does not raise or lower indices by means of the metric.

[81]Proofs, p. 2, [Hilbert 1915, p. 396].

function" H. The gravitational field equations are given as derivatives with respect to the metric, the electrodynamic field equations are given as derivatives with respect to the electromagnetic potential. Hence, the gravitational field equations appear in the form[82]

$$\frac{\partial\sqrt{g}H}{\partial g^{\mu\nu}} = \sum_k \frac{\partial}{\partial w_k}\frac{\partial\sqrt{g}H}{\partial g_k^{\mu\nu}} - \sum_{k,l}\frac{\partial^2}{\partial w_k \partial w_l}\frac{\partial\sqrt{g}H}{\partial g_{kl}^{\mu\nu}}, \tag{2}$$

where $\mu, \nu = 1, \ldots 4$, $g_k^{\mu\nu} \equiv \partial g^{\mu\nu}/\partial w_k$, $g_{kl}^{\mu\nu} \equiv \partial^2 g^{\mu\nu}/\partial w_k \partial w_l$ and factors of $1/2$ resp. $1/4$ are implicitly understood accounting for the symmetries in $\mu\nu$ and kl. Unfortunately, the part where the world function H in all probability was specified was later cut out of the proofs. In the published version it is given as the sum of a gravitational part K, depending only on the metric and its first and second derivatives, plus an electromagnetic part L, depending on the electromagnetic potential, its derivatives, and on the metric.[83] In all probability, also the excised piece of the proofs contained at least the specification of H as

$$H = K + L \tag{3}$$

with $K = K(g^{\mu\nu}, g_k^{\mu\nu}, g_{kl}^{\mu\nu})$ and $L = L(g^{\mu\nu}, q_s, q_{sl})$ where $q_{sl} = \partial q_s/\partial w_l$.[84] In any case, already in the proofs, Hilbert in a footnote insists that his world function *is* invariant whereas Einstein's is not.[85] Indeed, the Lagrangian functions presented by Einstein either in his second paper with Grossmann

[82] Proofs, p. 3. In [Hilbert 1915, p. 397] the equations were given with all terms moved to the left hand side and set equal to 0.

[83] [Hilbert 1915, p. 402].

[84] The quantities K and L are referred to immediately after the missing passage of [p. 8] of the proofs, and the splitting into $K + L$ is referred to later in the paper by an equation number which is missing exactly at that place. One possible reason for Hilbert's cutting out this piece would be that he wanted to paste it into some other manuscript in order to be spared the pains of copying the equations by hand. He had done a similar thing before in his notes for a lecture course on Euclidean geometry from 1898/99, when he pasted some of his axioms of connection ("Axiome der Verbindung") taken from [Hilbert 1895] into his manuscript notes, see SUB Cod. Ms. Hilbert 551, pp. 10f. Another such example is SUB Cod. Ms. Hilbert 549 (Notes for a lecture course on "Zahlbegriff und Quadratur des Kreises," winter 1897/98), pp. 38f. If the excised piece of the proofs had corresponded to the lines printed on the bottom of p. 402 of the published version, it would have contained the definitions of the Riemann curvature scalar and of the Ricci tensor as in eqs. (4) and (5).

[85] Proofs, p. 2, [Hilbert 1915, p. 396].

on the covariance properties of the *Entwurf* theory of spring 1914,[86] and in his major account of the *Formale Grundlage* of General Relativity of fall 1914,[87] and in his first November communication of 1915[88] are not invariant. Since Hilbert's world function *was* invariant and since Lagrangian differentiation with respect to the metric is a covariant operation, Hilbert's gravitational field equations already in the proofs *were* in any case generally covariant. And if the excised piece of the proofs contained the same specification as the published version,[89] K would have been specified as the Riemann curvature scalar,

$$K = \sum_{\mu\nu} g^{\mu\nu} K_{\mu\nu}, \tag{4}$$

where

$$K_{\mu\nu} = \sum_{\kappa} \left(\frac{\partial}{\partial w_\nu} \begin{Bmatrix} \mu\kappa \\ \kappa \end{Bmatrix} - \frac{\partial}{\partial w_\kappa} \begin{Bmatrix} \mu\nu \\ \kappa \end{Bmatrix} \right) + \sum_{k,\lambda} \left(\begin{Bmatrix} \mu\kappa \\ \lambda \end{Bmatrix} \begin{Bmatrix} \lambda\nu \\ \kappa \end{Bmatrix} - \begin{Bmatrix} \mu\nu \\ \lambda \end{Bmatrix} \begin{Bmatrix} \lambda\kappa \\ \kappa \end{Bmatrix} \right) \tag{5}$$

denotes the Ricci tensor and $\begin{Bmatrix} \mu\nu \\ \kappa \end{Bmatrix}$ are the Christoffel symbols of the second kind. With these specifications, the resulting field equations would differ from those of Einstein's second November memoir by a term $(1/2)g_{\mu\nu}K$ to be subtracted from the Ricci tensor. Nevertheless, even though Hilbert's field equations are not explicitly given in the early version in terms of the Ricci tensor and its contraction, Hilbert had probably realized that his theory in any case implied field equations which differed from the ones of Einstein's *Entwurf* theory or from those put forward in Einstein's first November communication. Indeed, the paper submitted to the Göttingen Academy initially was titled "Die Grund*gleichungen* der Physik." It was, however, soon changed to "Die Grund*lagen* der Physik."[90]

[86][Einstein and Grossmann 1914, p. 219].

[87][Einstein 1914, p. 1076].

[88][Einstein 1915b, p. 784].

[89][Hilbert 1915, p. 402].

[90]"Grundgleichungen der Physik" was the title of Hilbert's lecture to the Göttingen Mathematical Society of November 16. It was also the title under which Hilbert announced

3.2 The concept of energy

Hence, the restricted covariance of the theory in the proofs does not affect the gravitational field equations. But in the proofs the term "Grundgleichungen" does not refer to the gravitational field equations alone. As a consequence of a mathematical theorem, to be discussed in the next section, Hilbert saw that of the 14 field equations obtained by Lagrangian differentiation with respect to the metric and the electromagnetic potential, only $14-4=10$ were mutually independent. In the proofs, Hilbert then introduced four additional equations — of restricted covariance — in order to satisfy the principle of causality as Einstein had done in the *Entwurf* and as Hilbert initially thought necessary in order to guarantuee the determinate character of the differential equations in Cauchy's sense:

> Thus, if we want to preserve the determinate character of the fundamental equations of physics according to Cauchy's theory of differential equations, we must postulate four additional non-invariant equations which supplement [the gravitational and electrodynamic field equations][91]

Already in his letter to Einstein of November 14, Hilbert had talked about "the still missing 4 space-time equations."[92]

In order to arrive at such additional equations, Hilbert proceeded to define a concept of energy. Hilbert is not very explicit about the justification for his energy concept but what he does is quite clear. In his definition of the

his communication in the letter of invitation circulated among the Academy members between November 15 and the meeting of November 20, and it was the title under which his communication was initially entered into the "Journal" of the "Nachrichten", cp. note 73. Since, however, the first proofs, which I assume to represent Hilbert's manuscript of November 19, already have the title "Grundlagen der Physik," Hilbert must have changed the title probably with the manuscript which he gave to the printer, i.e., before or on November 19. The change in title may have been a reaction to Einstein's claim in his letter of November 18 that Hilbert's equations appear to be equivalent to the ones he had found and presented to the Academy in his first two November communications.

[91] "[...] so ist, wofern wir der Cauchyschen Theorie der Differentialgleichungen entsprechend den Grundgleichungen der Physik den Charakter der Bestimmtheit wahren wollen, die Forderung von vier weiteren zu [den Gravitations- und elektrodynamischen Feldgleichungen] hinzutretenden nicht invarianten Gleichungen unerläßlich." Proofs, p. 4. For a historical discussion of the early history of the Cauchy problem in General Relativity and Hilbert's role in it, see [Stachel 1992].

[92] Hilbert to Einstein, 13 November 1915, CPAE8, Doc. 140.

concept of energy he brings together techniques from invariant theory and the calculus of variations. He introduced what he called the "polarisation"[93] of the invariant worldfunction H with respect to an "arbitrary contragredient tensor" $h^{\mu\nu}$

$$J^{(h)} = \sum_{\mu,\nu} \frac{\partial H}{\partial g^{\mu\nu}} h^{\mu\nu} + \sum_{\mu,\nu,k} \frac{\partial H}{\partial g_k^{\mu\nu}} h_k^{\mu\nu} + \sum_{\mu,\nu,k,l} \frac{\partial H}{\partial g_{kl}^{\mu\nu}} h_{kl}^{\mu\nu}, \tag{6}$$

where the subscripts k and l to $h^{\mu\nu}$ denote partial derivatives with respect to w_k resp. w_l. Since polarisation is an invariant process, Hilbert argues, $J^{(h)}$ is an invariant. Hilbert now removes the partial derivatives on $h^{\mu\nu}$ by treating the expression $\sqrt{g}J^{(h)}$ "in the same manner as one treats in the calculus of variations the integrand of a variational problem if one wants to apply partial integration,"[94] thus obtaining

$$\sqrt{g}J^{(h)} = -\sum_{\mu\nu} H \frac{\partial \sqrt{g}}{\partial g^{\mu\nu}} h^{\mu\nu} + \sum_{\mu\nu} [\sqrt{g}H]_{\mu\nu}\, h^{\mu\nu} + D^{(h)}, \tag{7}$$

where $\left[\sqrt{g}H\right]_{\mu\nu} = 0$ is an abbreviation for the field equation (2), and $D^{(h)}$ a pure divergence term. The first term, obviously, arises from the fact that "polarisation" is applied to the invariant H, whereas the whole integrand $\sqrt{g}H$ must be varied. Without explicitly inserting the expression into the action integral, Hilbert thus effectively derived the gravitational field equations by variation of the action integral with respect to the metric, if only $h^{\mu\nu}$ be regarded as an arbitrary variation of the metric tensor. But at this point, he is content with a reformulation of the "polarized" integrand in terms of the field equations and a pure divergence term.

At this stage, Hilbert introduced the "symmetric contravariant tensor"

$$p^{\mu\nu} = \sum_s \left(g_s^{\mu\nu} p^s - g^{\mu s} p_s^{\nu} - g^{\nu s} p_s^{\mu}\right), \quad \left(p_s^j = \frac{\partial p^j}{\partial w_s}\right) \tag{8}$$

formed from an arbitrary vector p^{μ}.[95] In modern notation, the expression is readily identified as the Lie derivative of the metric tensor field $\underset{p}{\mathcal{L}} g^{\mu\nu} = -p^{(\mu;\nu)}$

[93] Proofs, p. 4. See [Kerschensteiner 1887, § 2.], for a discussion of the concept of polarisation in the theory of invariants. The letter from Emmy Noether to Fischer, quoted above (see note 69), indeed, suggests that [Kerschensteiner 1887] was a standard reference on invariant theory in Göttingen at the time.

[94] Proofs, pp. 4, 5.

[95] Note that p_s^{μ} is not obtained by lowering one index of $p^{\mu\nu}$, cp. note 80.

along the vector field p^μ and thus represents a variation of the metric $\delta g^{\mu\nu}$ which arises from a change of coordinates $x^\mu \to x^\mu + p^\mu(x)$.[96] Hilbert now substituted $p^{\mu\nu}$ for $h^{\mu\nu}$, and again removed all partial derivatives on p^s, except for first derivatives p^s_k. He thus arrived at

$$\sqrt{g}J^{(p)} = -\sum_{\mu\nu} H\frac{\partial\sqrt{g}}{\partial g^{\mu\nu}}p^{\mu\nu} + E + D^{(p)}, \tag{9}$$

where $D^{(p)}$ again is a divergence term and E was given explicitly as[97]

$$\begin{aligned} E = &\sum\left(H\frac{\partial\sqrt{g}}{\partial g^{\mu\nu}}g^{\mu\nu}_s + \sqrt{g}\frac{\partial H}{\partial g^{\mu\nu}}g^{\mu\nu}_s + \sqrt{g}\frac{\partial H}{\partial g^{\mu\nu}_k}g^{\mu\nu}_{sk} + \sqrt{g}\frac{\partial H}{\partial g^{\mu\nu}_{kl}}g^{\mu\nu}_{skl}\right)p^s \\ &-\sum\left(g^{\mu s}p^\nu_s + g^{\nu s}p^\mu_s\right)[\sqrt{g}H]_{\mu\nu} \\ &+\sum\left(\frac{\partial\sqrt{g}H}{\partial g^{\mu\nu}_k}g^{\mu\nu}_s + \frac{\partial\sqrt{g}H}{\partial g^{\mu\nu}_{kl}}g^{\mu\nu}_{sl} - g^{\mu\nu}_s\frac{\partial}{\partial w_l}\frac{\partial\sqrt{g}H}{\partial g^{\mu\nu}_{kl}}\right)p^s_k. \end{aligned} \tag{10}$$

Hilbert now defined the expressions e_s and e^l_s by rewriting E as

$$E = \sum_s e_s p^s + \sum_{s,l} e^l_s p^s_l. \tag{11}$$

Both e_s and e^l_s are well-defined by eq. (10), and, as Hilbert noted, e_s is given as a total derivative with respect to w_s,

$$e_s = \frac{d^{(g)}\sqrt{g}H}{dw_s}, \tag{12}$$

[96]Hilbert only introduced $p^{\mu\nu}$ as a symmetric contravariant tensor without mentioning that it represents a Lie derivative or a variation of the metric. However, Klein, in a first note from January 1918 on Hilbert's first communication, refers in this context to Lie's "numerous relevant publications" ("zahlreiche einschlägige Veröffentlichungen," [Klein 1917, p. 471]). In a second note on Hilbert's paper from July 1918, Klein nevertheless found it necessary to elaborate on the meaning of $p^{\mu\nu}$ by an explicit calculation in an extra introductory paragraph ([Klein 1918, p. 173]).

[97]It is at this point that the proofs which otherwise are of a fairly finished character display some irregularities which may indicate a certain preliminary nature of the consideration. First, the sentence introducing this expression is grammatically incomplete. Second, Hilbert corrected the factor p^s multiplying the first term from a misprinted p_s and put an exclamation mark in the margin (proofs, p. 5). Third, in the remainder of the proofs, Hilbert mistakenly referred to his equation (10), which is our (9), where clearly his equation (9), which is our (10), was meant.

where the superscript (g) indicates that the electromagnetic potentials are to be ignored in forming the derivative.

Hilbert called the expression E the energy form ("Energieform"), justifying this name by pointing out two properties. First, he noted that a comparison of $\sqrt{g}J^{(h)}$ as given in eq. (7), where $p^{\mu\nu}$ is substituted for $h^{\mu\nu}$, with $\sqrt{g}J^{(p)}$ as given in eq. (9) shows that E may be expressed as a sum of derivatives with respect to w_s,

$$E = \left(D^{(h)}\right)_{h=p} - D^{(p)}, \tag{13}$$

if the gravitational field equations hold and hence becomes a divergence.[98]

The discussion of the second property, unfortunately, was mutilated by cutting away part of the sheet on which it was given. But the argument can nevertheless be reconstructed. Looking at E as given in (11) Hilbert noted that if one were to go one step further and also remove the derivative with respect to w_l in p_l^s the divergence equation "corresponding to the energy theorem in the old theory",

$$\sum_l \frac{\partial e_s^l}{\partial w_l} = 0 \tag{14}$$

can only hold if and only if the four quantities e_s vanish, i.e., if and only if

$$e_s = 0. \tag{15}$$

The latter two equations are not generally covariant. The validity of eq. (14) and hence the validity of eq. (15) was stipulated by a third axiom, and the four equations (15) complemented the ten gravitational field equations (2) to a system of 14 equations for the 14 potentials $g^{\mu\nu}$, q_s. It is these 14 equations, (2) and (15), which Hilbert in the proofs called the "system of fundamental equations of physics."[99]

This part of the proofs, i.e., the derivation of the energy concept was rewritten for the published version. The third axiom, introduced in the proofs, postulated a restriction of the covariance of the theory. Its stipulation introduced a distinction between space-time coordinates and world parameters, the former being those coordinates for which the non-generally covariant energy equations (14) hold. This third axiom was later dropped in the published version.

[98]"erhält Divergenzcharakter," proofs, p. 6.
[99]Proofs, [p. 7].

3.3 Theorems about the invariant variational problem

Both the proofs and the published paper present three mathematical theorems, all three of them drawing consequences from the invariance of a Lagrangian function and the action integral with respect to arbitrary coordinate transformations.

The first theorem is a special case of a general theorem about invariant variational problems proven three years later by Emmy Noether and known as Noether's (second) theorem.[100] In Hilbert's communication it is formulated for an invariant "world function" J of the four "world parameters" which depends on n variables and their derivatives. The theorem then asserts that if the n Lagrangian variational equations are formed, four mutually independent linear combinations of these n differential equations and their total derivatives are always satisfied, i.e., vanish identically. So far, the theorem is a special case of Noether's theorem. But Hilbert went one step further by asserting that the theorem implied that four of the n differential equations are always a consequence of the remaining $n-4$ equations "in the sense" that those linear combinations hold. The proof of this theorem was given neither in the proofs nor in the published version but announced for another occasion.[101]

While this first theorem does not give an explicit construction of these identities and, in fact, only asserts their existence, the other two mathematical theorems of Hilbert's first note are formulated more explicitly. Hilbert did not prove these theorems in the proof sheets either and introduced their formulation by simply saying he would make use of them,[102] thus suggesting that they were either well-known or trivial. In the published version, however, he did prove these two other theorems and introduced their formulation by saying he would now "establish" those two theorems.[103]

For the second theorem, Hilbert added an alternative formulation in his published version which makes it more lucid. In this alternative formulation, the theorem asserts that, given an invariant Lagrangian J depending on the metric components $g^{\mu\nu}$, its first and second derivatives, and on the

[100][Noether 1918, p. 239].

[101]Proofs, pp. 2f, [Hilbert 1915, pp. 397f]. For a discussion of this theorem in Hilbert's note, see [Vizgin 1994, pp. 54–69]. Vizgin does not note, however, that Hilbert's formulation of the theorem actually goes beyond Noether's theorem, cp. also footnote 120 below.

[102]"Des Weiteren benutzen wir zwei mathematische Theoreme, ..." proofs, [p. 8].

[103]"Des Weiteren stellen wir zwei mathematische Theoreme auf, ..." [Hilbert 1915, p. 398].

electromagnetic potentials q_s and its first derivatives, the following identity

$$\sum_s \frac{\partial J}{\partial w_s} p^s = PJ, \tag{16}$$

holds,[104] where

$$P = P_g + P_q, \tag{17}$$

$$P_g = \sum_{\mu,\nu,l,k} \left(p^{\mu\nu} \frac{\partial}{\partial g^{\mu\nu}} + p_l^{\mu\nu} \frac{\partial}{\partial g_l^{\mu\nu}} + p_{lk}^{\mu\nu} \frac{\partial}{\partial g_{lk}^{\mu\nu}} \right), \tag{18}$$

$$P_q = \sum_{l,k} \left(p_l \frac{\partial}{\partial q_l} + p_{lk} \frac{\partial}{\partial q_{lk}} \right), \tag{19}$$

and

$$p_l = \sum_s (q_{ls} p^s + q_s p_l^s), \tag{20}$$

and $p_{lk} = \partial p_l / \partial w_k$. In eq. (16) the derivative with respect to w_s is, in fact, to be understood as a total derivative, taking into account that J depends on $g^{\mu\nu}$, $g_l^{\mu\nu}$, $g_{lk}^{\mu\nu}$, q_l, and q_{lk} but not explicitly on the "world parameters" w_s. In the proofs, the theorem is only stated in a form which is obtained if the terms implied by the left hand side of (16) are subtracted from the right hand side and the remaining terms set equal to 0.[105]

The third theorem is effectively a derivation of generalized contracted Bianchi identities for a Lagrangian J depending only on the metric components and its first and second derivatives. Using the definition (8) of $p^{\mu\nu}$ and introducing new notation,

$$\sum_{\mu\nu} [\sqrt{g}J]_{\mu\nu}\, p^{\mu\nu} = \sum_{s,l} (i_s p^s + i_s^l p_l^s), \tag{21}$$

with

$$i_s = \sum_{\mu\nu} [\sqrt{g}J]_{\mu\nu}\, g_s^{\mu\nu}, \tag{22}$$

$$i_s^l = -2 \sum_{\mu} [\sqrt{g}J]_{\mu s}\, g^{\mu l}, \tag{23}$$

[104] ibid.
[105] Proofs, [p. 8].

the theorem asserts that the equation

$$i_s = \sum_l \frac{\partial i_s^l}{\partial w_l} \tag{24}$$

"holds identically for all arguments, i.e., the $g^{\mu\nu}$ and their derivatives."[106]

3.4 Generalized electrodynamics and Mie's theory

Using his theorems II and III, Hilbert pointed out two features of his general variational framework which concern the relation of the theory to special relativistic electrodynamics and, in particular, to Mie's theory as given in Born's interpretation. In order to appreciate Hilbert's arguments, it is useful to briefly summarize the main points of Born's reinterpretation of Mie's theory.[107] Mie had formulated his theory in terms of a variational principle, introducing a Lorentz-covariant Hamiltonian depending on the fields and the electromagnetic four-potential. Born had generalized this ansatz by postulating a general Lorentz-covariant Lagrangian Φ depending on dynamic variables $u_\alpha = u_\alpha(x_1, x_2, x_3, x_4)$ and their derivatives $a_{\alpha\beta} \equiv \partial u_\alpha / \partial x_\beta$ in a four-dimensional Minkowski spacetime with coordinates x_α. The dynamics of the general theory was governed by the variational principle

$$\delta \int \Phi(a_{ij}; u_i) dx_1 dx_2 dx_3 dx_4 = 0, \tag{25}$$

which yielded general Euler-Lagrange equations

$$\sum_\gamma \frac{\partial X_{\beta\gamma}}{\partial x_\gamma} - X_\beta = 0, \tag{26}$$

where

$$\frac{\partial \Phi}{\partial a_{\alpha\beta}} = X_{\alpha\beta}, \qquad \frac{\partial \Phi}{\partial u_\alpha} = X_\alpha. \tag{27}$$

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[106]Proofs, p. 9, [Hilbert 1915, p. 399]. If we take J to be the Riemann curvature scalar $K = \sum g^{\mu\nu} K_{\mu\nu}$, as in eqs. (4), (5) above, then $\sum_{\mu\nu} \left[\sqrt{g} J\right]_{\mu\nu}$ is equivalent to the tensor $\sqrt{g} G_{\mu\nu} = \sqrt{g}(K_{\mu\nu} - (1/2) g_{\mu\nu} K)$, and eq. (24) expresses the vanishing of its covariant divergence, $G^l_{s;l} = 0$. See also [Klein 1917, p. 472].

[107][Born 1914].

Interpreting the dynamical variables u_α as displacements of a four-dimensional continuum, the ansatz could be specialized to special relativistic elasticity theory. In this case Φ would depend only on 6 combinations e_{ij} $(i, j = 1, 2, 3)$ of the $a_{\alpha\beta}$, representing the deformation quantities introduced by Born in his definition of relativistic rigidity.[108] Alternatively, the dynamic variables u_α may be regarded as the components of the electromagnetic four-potential. This would give the specialization corresponding to Mie's theory. In this case Φ only depended on the antisymmetric combinations $a_{\alpha\beta} - a_{\beta\alpha}$, representing the components of the electromagnetic field tensor. The main point of Born's paper was the identification of Mie's energy-momentum tensor in this generalized framework. In the general Lagrangian formulation, the Euler-Lagrange equations (26) entail conservation laws of the form

$$\frac{\partial \Phi}{\partial x_\alpha} = \sum_\gamma \frac{\partial}{\partial x_\gamma}(\sum_\beta X_{\beta\gamma} a_{\beta\alpha}), \tag{28}$$

if Φ does not depend explicitly on the coordinate x_α.[109] These conservation laws then justify the definition of the energy momentum tensor in terms of the Lagrangian Φ as

$$T_{\alpha\beta} = \Phi\delta_{\alpha\beta} - \sum_\gamma a_{\gamma\alpha} X_{\gamma\beta}, \tag{29}$$

(now known as the "canonical" energy-momentum tensor) since it could then be written in the compact notation of four-dimensional vector calculus as

$$\mathrm{Div}\, T = 0. \tag{30}$$

The point of Born's paper, discussed in its final paragraph, was to show that the energy-momentum tensor of Mie's theory is not simply given by eq. (29) since it has to depend only on the antisymmetric combinations $a_{\gamma\alpha} - a_{\alpha\gamma}$. Rather it is obtained by adding a tensor $a_{\alpha\gamma}X_{\gamma\beta} - u_\alpha X_\beta$ with vanishing four-divergence such that it is the (symmetric)[110] tensor

$$S_{\alpha\beta} = \Phi\delta_{\alpha\beta} - \sum_\gamma (a_{\gamma\alpha} - a_{\alpha\gamma}) X_{\gamma\beta} - u_\alpha X_\beta \tag{31}$$

[108] [Born 1909c, pp. 10, 15], [Herglotz 1911].

[109] This assumption distinguishes Mie's theory from the usual Maxwell theory with charges and currents as external sources as given by the usual Lorentz electron theory. This theory can formally be included into the general framework by letting Φ depend on the external sources, however, then Φ would explicitly depend on the space-time variables.

[110] [Mie 1912a, p. 533], [Born 1914, p. 35].

that represents the energy-momentum of Mie's theory.

It is to this reformulation of Mie's electrodynamics by Born that Hilbert referred to in the following part of his paper. By his first axiom, Hilbert emphasized, in the proofs as well as in the published version, that his world function H would not only depend on the metric tensor and its first and second derivatives but also on the electromagnetic potentials q_s and its first derivatives q_{sl}. In fact, Hilbert emphasized in a footnote that while the general use of a variational principle was a characteristic of Mie's theory, the particular use of the arguments q_s and q_{sk} was introduced by Born.[111]

Hilbert now pointed out two consequences following from the assumption of general invariance for H. From his Theorem II, Hilbert deduced the following relation for the electromagnetic part L of the world function

$$\sum_{\mu,\nu,m} \frac{\partial L}{\partial g^{\mu\nu}}(g^{\mu m}p^{\nu}_{m} + g^{\nu m}p^{\mu}_{m}) - \sum_{s,m} \frac{\partial L}{\partial q_s} q_m p^m_s - \sum_{s,k,m} \frac{\partial L}{\partial q_{sk}}(q_{sm}p^m_k + q_{mk}p^m_s + q_m p^m_{sk}) = 0, \qquad (32)$$

since, by assumption, L does not depend on the derivatives of the metric. By setting the coefficient in front of p^m_{sk} equal to 0, Hilbert then deduced that L could, in fact, only depend on the antisymmetrized derivatives

$$M_{ks} = q_{sk} - q_{ks}. \qquad (33)$$

Thus, Hilbert had shown that it followed from his second axiom, i.e., from the postulate of general covariance, that the world function $H = K + L$ can indeed only depend on the antisymmetrized derivatives of the electromagnetic potential, i.e., on the field tensor, although the electromagnetic part L of world function H initially was introduced as a function depending on all derivatives of q_s.[112] Hence Hilbert had shown that, in the generalized framework, this feature of Maxwellian electromagnetic theory was a consequence of the invariance of the world function with respect to arbitrary coordinate transformations:

> This result, which first gives the special character of the Maxwell equations, arises here essentially as a consequence of the general

[111] Proofs, p. 2, [Hilbert 1915, p. 396].

[112] Proofs, p. 10, [Hilbert 1915, p. 403].

invariance, i.e., on the basis of axiom II.[113]

The second consequence of the postulate of general invariance directly referred to Mie's energy-momentum tensor. Again referring to eq. (32), Hilbert noted that setting the coefficient of p^{ν}_{m} equal to 0 and taking into account that L depends only on the field variables M_{ks} as defined in (33) one obtains for the electromagnetic energy the expression

$$-2\sum_{\mu}\frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}}g^{\mu m}=\sqrt{g}\left\{L\delta^{m}_{\nu}-\frac{\partial L}{\partial q_m}q_\nu-\sum_{s}\frac{\partial L}{\partial M_{ms}}M_{\nu s}\right\}. \tag{34}$$

Comparing this expression with the corresponding expression (31) of Born's theory, Hilbert noted that Mie's energy tensor is obtained in his generally covariant, variational framework by generally covariant operations and subsequent specialization to the case of special relativity,[114] an insight obtained at least by mid-October and already then communicated to Sommerfeld and an insight which Hilbert had taken great pleasure in ("Hauptvergnügen"), as he reports to Einstein in his correspondence of November 13. Nowadays, the energy tensor of any theory is obtained by variation of the matter Lagrangian with respect to the metric, but for Hilbert this result was intimately linked with his understanding of the theory as a generalization of Mie's electrodynamic field theory of matter. In his paper, Hilbert summarized this insight in the following words:

> Hence, Mie's electromagnetic energy tensor is nothing but that generally covariant tensor which is obtained by differentiation of the invariant L with respect to the gravitation potentials $g^{\mu\nu}$ in that limiting case [of special relativity][115]

and immediately added that this was

[113]"Dieses Resultat, durch welches erst der Charakter der Maxwellschen Gleichungen bedingt ist, ergibt sich hier wesentlich als Folge der allgemeinen Invarianz, also auf Grund von Axiom II." [Hilbert 1915, p. 403]. In the proofs, p. 10, the explanatory half-sentence "durch welches ... bedingt ist" is missing.

[114]Proofs, p. 10, [Hilbert 1915, p. 404].

[115]"[...] der Mie'sche elektromagnetische Energietensor ist also nichts anderes als der durch Differentiation der Invariante L nach den Gravitationspotentialen $g^{\mu\nu}$ entstehende allgemeine invariante Tensor beim Übergang zu jener Grenze [der speziellen Relativitätstheorie] ..." proofs, p. 10, [Hilbert 1915, p. 404].

> — a fact which pointed me in the first place to the necessary and close relation between Einstein's general theory of relativity and Mie's electrodynamics and which convinced me of the correctness of the theory developed here.[116]

3.5 Gravitation, electromagnetism, and the theory of matter

The first mathematical theorem of Hilbert's paper, as discussed above, asserted the existence of four independent identities between the fourteen field equations obtained by Lagrangian differentiation of the action integral. This mathematical theorem was interpreted by Hilbert as implying that the generalized Maxwell equations may be regarded as a consequence of the gravitational field equations in the sense that the four linear combinations of the Maxwell equations and their total derivatives identically vanish. Thus Hilbert wrote

> As a consequence of that mathematical theorem we can immediately assert that, in the sense indicated, the electrodynamical phenomena are effects of gravitation.[117]

and, adding even more emphasis to this feature of his theory, he called this insight into the relation between electromagnetism and gravitation the "Leitmotiv" of his theory.[118] However, although Hilbert seemed to believe that the validity of the gravitational field equations implied the validity of the generalized Maxwell equations, he concretized his assertion by deriving the precise mathematical form of those four identities in the final part of his paper.

[116]"— ein Umstand, der mich zum ersten Mal auf den notwendigen engen Zusammenhang zwischen der Einsteinschen allgemeinen Relativitätstheorie und der Mie'schen Elektrodynamik hingewiesen und mir die Überzeugung von der Richtigkeit der hier entwickelten Theorie gegeben hat." proofs, p. 10, [Hilbert 1915, p. 404].

[117]"[...] wir können unmittelbar wegen jenes mathematischen Satzes die Behauptung aussprechen, daß in dem bezeichneten Sinne die elektrodynamischen Erscheinungen Wirkungen der Gravitation sind." Proofs, p. 3, [Hilbert 1915, p. 397].

[118]Proofs, p. 2, [Hilbert 1915, p. 396].

Starting from the electromagnetic energy as given in (34) as derivative of $\sqrt{g}L$ with respect to $g^{\mu\nu}$, Hilbert used the gravitational field equations

$$[\sqrt{g}K]_{\mu\nu} + \frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}} = 0, \tag{35}$$

which hold for the ansatz $H = K + L$,[119] and the definition (23) of i_s^l in order to express the left hand side of (34) as

$$-2\sum_{\mu} \frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}} g^{\mu m} = -i_\nu^m. \tag{36}$$

In order to invoke the contracted Bianchi identity $i_\nu = \sum \partial i_\nu^m / \partial w_m$ of his theorem III, Hilbert formed the coordinate divergence of the right hand side of (34) and obtained

$$i_\nu = \sum_m \frac{\partial}{\partial w_m}\left(-\sqrt{g}L\delta_\nu^m + \frac{\partial\sqrt{g}L}{\partial q_m} q_\nu + \sum_s \frac{\partial\sqrt{g}L}{\partial M_{sm}} M_{s\nu}\right), \tag{37}$$

and by further transformation of this expression, introducing the Lagrangian derivatives of the generalized Maxwell equations

$$[\sqrt{g}L]_m = \frac{\partial\sqrt{g}L}{\partial q_m} - \sum_s \frac{\partial}{\partial w_s}\frac{\partial\sqrt{g}L}{\partial q_{ms}} \tag{38}$$

and by making use of the antisymmetry of M_{sm}, the definition (22) of i_ν, and the relation $\partial L/\partial q_{sm} = -\partial L/\partial q_{ms}$, Hilbert finally arrived at the four identities

$$\sum_m \left(M_{m\nu}\,[\sqrt{g}L]_m + q_\nu \frac{\partial}{\partial w_m}\,[\sqrt{g}L]_m\right) = 0. \tag{39}$$

He summarized the result by repeating his claim that

> the gravitational equations [...] entail indeed the four mutually independent linear combinations (39) of the electrodynamic fundamental equations [...] and their first derivatives[120]

[119] Cp. footnote 84 above.

[120] "[...] aus den Gravitationsgleichungen [...] folgen in der Tat die vier von einander

and emphasized:

> This is the full mathematical expression of the above asserted proposition about the character of electrodynamics as a phenomenon following from gravitation.[121]

Finally, the freedom to specify the world function by non-linear so long as invariant terms of the electromagnetic potential in the sense of Mie opened up the possibility that non-linear generalizations of Maxwell's equations may be obtained, the solutions of which would properly describe the electron. It was the computation of the electron in this sense that Hilbert hoped to achieve before he wrote down his theory for print, and which he now left for the second part of his paper.

> Further elaboration and, in particular, the application of my basic equations to the fundamental questions of electricity theory I defer to a later communication.[122]

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unabhängigen linearen Kombinationen (39) der elektrodynamischen Grundgleichungen [...] und ihrer ersten Ableitungen." Proofs, p. 12, [Hilbert 1915, p. 406]. As John Stachel pointed out in the discussion at the Göttingen workshop, the relations (39) do not imply the electromagnetic field equations themselves. From the second term it follows that for $q_\nu \neq 0$ and given field equations $\left[\sqrt{g}L\right]_m = 0$ on an initial hypersurface $w_4 = \text{const}$, one has $\partial\left[\sqrt{g}L\right]_4/\partial w_4 = 0$, but no restrictions for $\partial\left[\sqrt{g}L\right]_i/\partial w_4$, $i = 1, 2, 3$ are implied. Therefore, even if the electromagnetic field equations hold on an initial hypersurface, they will not continue to hold automatically off it as a consequence of the identities. The second term in eq. (39) vanishes if L is gauge invariant. In this case the electromagnetic field equations do follow algebraically if $M_{m\nu}$ has an inverse. The relation was derived in this form in [Klein 1917, p. 473] who had started from an electromagnetic part $L = \alpha M_{\nu\mu}M_{\sigma\rho}(g^{\mu\rho}g^{\nu\sigma} - g^{\mu\sigma}g^{\nu\rho})$. Therefore, Klein can agree with Hilbert's claim that the Maxwell equations follow from the gravitational field equations.

[121]"Dies ist der ganze mathematische Ausdruck der oben allgemein ausgesprochenen Behauptung über den Charakter der Elektrodynamik als einer Folgeerscheinung der Gravitation." ibid. [Hilbert 1915, p. 406] changed "ganze" to "genaue."

[122]"Die genauere Ausführung sowie vor Allem die spezielle Anwendung meiner Grundgleichungen auf die fundamentalen Fragen der Elektrizitätslehre behalte ich späteren Mitteilungen vor." Proofs, p. 1, [Hilbert 1915, p. 395]. See also Hecke to Hilbert, 7 March 1916, "When will you publish the electron?" ("Wann publizieren Sie das Elektron?") SUB Cod. Ms. Hilbert 141/7.

4 A speculative reconstruction

Given these characteristics of Hilbert's theory which are already found in the first proofs, the following scenario of Hilbert's path towards his unified field theory of electromagnetism and gravitation, including generally covariant gravitational field equations and generalized Maxwell equations in the sense of Mie, is suggested. The scenario would be the following: being interested in fundamental theories of physics, Hilbert studied Mie's theory as early as 1913 as a novel and attractive field theory of matter. An important step was Born's work of late 1914 on Mie's theory, which reinterpreted Mie's variational ansatz in terms of special-relativistic continuum mechanics, pointing out formal relations to Herglotz's special-relativistic elasticity theory, which Hilbert knew well. Hilbert had also heard about Einstein's and Grossmann's theory as early as December 1913 through a talk at the Göttingen Mathematical Society. Just as in the case of Mie's theory, Hilbert must have had a genuine interest in attempts at generalizing the special theory of relativity, for Hilbert the heritage of his friend Minkowski's geometric reinterpretation of the Lorentz transformations. When Hilbert then succeeded in inviting Einstein to come to Göttingen he was intrigued by the latter's presentation. As Einstein found, Hilbert and the other Göttingen mathematicians were completely convinced by the theory.[123]

Thus, when Hilbert went away for his summer vacation to Switzerland and Rügen, he had the following situation to ponder on. Both Mie's theory, in particular as reinterpreted by Born, and Einstein's theory of 1914 started from a variational principle with a Lagrangian of a certain covariance group. Mie's theory was invariant with respect to the orthogonal group of special relativity. The mathematics of Einstein's theory was set up to be invariant with respect to general diffeomorphisms but in his 1914 exposition of the *Formale Grundlage der allgemeinen Relativitätstheorie* Einstein had expounded the hole argument justifying the necessity of restricting the covariance of the theory and elaborated an intricate mathematical argument for the unique specification of the correct Lagrangian in accordance with his so-called "adapted" coordinate systems. The argument was based on a rather pedestrian treatment of the covariance properties of the action integral, distinguishing between somewhat hybrid variations restricted by Einstein's concept of "justified" coordinate transformations. Now, invariant theory and the

[123][Pais 1982, p. 259], CPAE8, Doc. 110, cp. note 47.

calculus of variations were fields of Hilbert's expertise, and when Einstein presented his considerations in July 1915 in Göttingen, he found that everything was understood, including the details. Nevertheless, Einstein's mathematical treatment of adapted coordinate systems must have appeared awkward to Hilbert just as they had appeared fishy to the mathematician Levi-Civita.[124]

Mie's theory on the other hand had other problems. For one, Mie's theory of gravitation might not have been too attractive. Also, Mie had specified his world function by adding a term of sixth power in the potential and had explicitly integrated the resulting differential equation.[125] The result was unsatisfactory to Mie because the resulting electrons would neither be indivisible nor stable. Mie's electrons would tend to attract each other and then simply merge into a single huge lump of charge.[126] However, his theory formulated a research program by posing the problem of finding a satisfactory Lagrangian function. The research program was formulated by Born as follows:

> In earlier times, one formulated the goal of the mechanical explanation of nature, to derive all physical and chemical properties of matter by assuming a Lagrangian function Φ for the interaction of the atoms. In the same way Mie now poses the problem of chosing his "world function" Φ such that from it follows on the basis of his differential equations the existence of the electron and of the atoms, as well as all their interactions. I should like to see this problem as the mathematical content of that program which regards the establishment of an "electrodynamic worldview" as the aim of physics.[127]

[124]For a discussion of Levi-Civita's objections to Einstein's proof of the covariance properties of the field equations in [Einstein 1914] expressed in a correspondence between him and Einstein in the spring of 1915, see [Cattani and De Maria 1989]. Whatever Hilbert's objections to Einstein's theory were, they referred in any case to Einstein's way of handling variations of the metric, see Einstein to Hilbert, 30 March 1916, CPAE8, Doc. 207.

[125][Mie 1912b, pp. 18ff].

[126][Mie 1912b, p. 38].

[127]"Wie man früher als Ziel der mechanischen Naturerklärung hinstellte, durch Annahme einer Lagrangeschen Funktion Φ für die Wechselwirkung der Atome alle physikalischen und chemischen Eigenschaften der Materie abzuleiten, so stellt jetzt Mie die Aufgabe, seine 'Weltfunktion' Φ so zu wählen, daß daraus auf Grund seiner Differentialgleichungen sowohl die Existenz des Elektrons und der Atome, als auch die Gesamtheit ihrer Wechselwirkungen hervorgehen. Diese Forderung Mies möchte ich als den mathematischen Inhalt jenes Programms ansehen, das die Aufstellung eines 'electromagnetischen Weltbildes' als

And Hilbert, who knew well both the relation between a variational ansatz and the implied differential equation and the complexity of non-linear differential equations, regarded Mie's research program, as formulated here by Born, as the real challenge of theoretical physics.

One of Hilbert's first ideas might have been to generalize the covariance of Mie's theory and to integrate it into a common mathematical framework of both Mie's and Einstein's theories. Mathematically, something like the metric tensor introduced by Einstein would also show up if one simply allowed for general coordinate transformations in Mie's theory. Why not let Mie's world function depend on the metric components as well, and add it to the Lagrangian of Einstein's theory? Using generally covariant operations, one might then see whether Mie's results are obtained in the special relativistic limit, which would give concrete physical meaning to the generalized theory. This indeed proved to be the case for Mie's energy tensor. Would it also hold for the electron? Hilbert might then have investigated general properties of the resulting differential equations, drawing on his knowledge of partial differential equations and on knowledge about what we would now call the Lie derivative. This might have given him the insight that covariance with respect to diffeomorphisms yielded additional relations between the field equations, a result which immediately suggested that the ten gravitational field equations and the Maxwell equations were not independent.

These are vague considerations, of course. The Hilbert archive contains a number of disparate calculations, which are hard to identify or even only to date. There is opportunity for future research. In any case, Hilbert obtained most of the characteristic features of his first note already in the fall of 1915, with the exception of the explicit form of the field equations in terms of the Ricci tensor and the realization that the restrictive implications of energy conservation are but a remnant dark spot of Einstein's previous misconceptions.

5 Aftermath

Five days after Hilbert had submitted his communication to the Göttingen Academy, Einstein presented a fourth communication to the Berlin Academy.[128] In this communication Einstein presented his final field equations, the Ein-

Ziel der Physik hinstellt." [Born 1914, pp. 24].

[128] [Einstein 1915a].

stein equations where a trace term was added to the energy-momentum tensor. The introduction of the trace term had two decisive consequences. First, the requirement of energy conservation no longer implied restrictions on the permissible coordinates. The theory now was fully covariant. A distinction, in Hilbert's terms, between "world parameters" and "space-time coordinates" was no longer necessary. Second, the gravitational field equations no longer implied that the trace of the energy-momentum tensor of matter had to vanish. Hence, the field equation no longer implied anything about the constitution of matter.

We don't know whether or when Einstein sent copies of his final communication to Hilbert, but we may assume that he did, just as he had sent his first communication to Hilbert, and in any case Einstein's final communication was available in print by December 2.[129]

Hilbert, therefore, had occasion to think about his own theory in the light of Einstein's last November communication. And in any case, he continued working on his theory. Thus, roughly a week later, on November 30, he and Caratheodory gave a talk to the Göttingen Mathematical Society on invariant theory.[130] And four days later, on December 4, Hilbert presented a second communication on the "Foundations of Physics" to the Göttingen Academy. About the content of this second communication nothing is known since its publication was postponed, at least for the time being.[131] On that Academy meeting of December 4, Hilbert also presented a note by Emmy Noether with the title "Krümmungsinvarianten im mehrdimensionalen Raume" for publication in the Academy's *Nachrichten.* We don't know either what the point of this paper was since it was later officially withdrawn[132] and the manuscript appears to be lost. It is known, however, that on the same

[129]CPAE6, p. 244.

[130]JDM 24 (1915), p. 111.

[131]GM 1916, p. 6. The "Journal" for the "Nachrichten" (GAA Scient 66, Nr. 2, cp. note 73), entry Nr. 731, says "wiederholt in No 739." Entry Nr. 739 of the "Journal" lists a communication by Hilbert, titled "Grundlagen der Physik Zweite Mitteilung," presented on 26 February. The date of 2 March 1916 of this entry in the column "Zum Druck" was deleted and in the last column its says "=No. 731, 759." Entry Nr. 759 finally is Hilbert's second communication: "Grundlagen der Physik (Zweite Mitteilung)," presented on 23 December 1916, sent to the printer on 29 January 1917, and published as [Hilbert 1917].

[132]The "Journal" for the "Nachrichten" (GAA Scient 66, Nr. 2, cp. the previous note and note 73) in this case says "zurückgezogen," see entry Nr. 733, and indeed a paper with this title was never published by Emmy Noether; see also GM 1916, p. 6 where Noether's paper was also announced.

day, December 4, 1915, Hilbert wrote to the Prussian minister on behalf of Noether's habilitation. In this letter he wrote:

> ... also the mathematical-physical developments (Einstein's theory of gravitation, theory of time and space) are presently moving towards an unforeseen point of culmination; and in this matter Miss Noether is my most successful collaborator ...[133]

The next day, December 5, Hilbert, Klein, Voigt, Runge, and Wiechert wrote a proposal to the Göttingen Academy suggesting that Einstein be elected as a corresponding member.[134] Two days later, on December 7, Hilbert and Caratheodory continued their lecture on invariant theory in the Göttingen Mathematical Society.[135]

With Hilbert's communication in press and Einstein's final communication already published, the hectic feverishness of November now abated somewhat. It seems that Hilbert took the occasion of Einstein's election as a corresponding member of the Göttingen Academy, which took place on the Academy meeting of December 18, to inform him about his election in a personal letter before the official notice.[136] It is in response to this letter that

[133]"... auch die mathematisch-physikalischen Fortschritte (Gravitationstheorie von Einstein, Theorie von Zeit und Raum) streben einem ungeahnten Kulminationspunkt gerade gegenwärtig zu; und da habe ich hier Frl. Emmy Noether als erfolgreichste Mitarbeiterin ..." Hilbert to the Prussian Ministry, 4 December 1915, quoted in [Tollmien 1991, p. 22].

[134]See note 136 below.

[135]JDM 24 (1915), p. 111.

[136]The chronology of Einstein's election as a corresponding member of the Göttingen Academy is as follows: on November 22, the secretary of the mathematical-physical class, Ehlers, had sent out an invitation to an "unverbindliche Vorbesprechung" on Saturday, November 27, following a request by Klein ("Nach einem Wunsche des Herrn F. Klein" (GAA Pers 10.3, 25). It was in all probability Klein who took the initiative about the elections to the Academy in December 1915; see notes about the Academy's members and a list of possible candidates in SUB Cod. Ms. Klein III F, sheets 9–11, dated 29 November 1915. The proposal for the election of Einstein ("einer der tiefsten und erfolgreichsten Forscher im Gebiete der theoretischen Physik") as a corresponding member was dated December 5 and was signed by Hilbert, Klein, Runge, Voigt, and Wiechert (GAA Pers 20, 913). On December 8, Ehlers invited the members of the class to the meeting on Saturday, December 11, announcing that it had been suggested to elect P. Debye and H. Stille as full ("ordentlich") members, G. Cantor as external ("auswärtig") member, and A. Einstein as well as P. Koebe as corresponding ("korrespondierend") members (GAA Pers 10.3, 25). The final election then took place at the joint meeting of both classes on December 18 (GAA Chron 2.1, Vol. 6, 413). The official notice was sent out on December 22 (GAA

Einstein himself wrote a conciliatory response to Hilbert after having somewhat settled his own emotional turmoil around his November breakthrough, writing

> It is objectively a pity if two guys who have somewhat worked their way free of this shabby world don't take pleasure in one another.[137]

After having been forced to a somewhat premature write up of his results, Hilbert now reconsidered the calculations in the first set of proofs for his communication, in particular those about the energy theorem. It probably took Hilbert some time but eventually he found that he could indeed derive an energy expression which also gave Mie's energy tensor in the limit of special relativity but which satisfied a conservation law that was not of restricted covariance. The latter feature was in agreement with Einstein's fourth and final commmunication of November 25 which contained the final field equations of Einstein's theory. The additional trace term introduced in that communication to the gravitational field equations implied that energy-momentum conservation no longer posed a restriction on the admissible coordinates. Now Hilbert also found an energy expression which, together with his field equation, was identically conserved without imposing additional constraints. It is not clear when Hilbert achieved this insight but on January 25, 1916, he

Chron 8, Vol. 6). Einstein acknowledged his election in a letter to the Academy dated December 23 (GAA Pers 20, 416).

In 1923, Born suggested that A. Einstein and N. Bohr be elected as external members of the Academy (GAA Pers 19, 498). Einstein's letter of acknowledgment of 17 October 1923 contains the following amusing postscript:

> "Ich vermute, dass ich schon Mitglied der 'Gesellschaft der Wissenschaften in Göttingen' bin, da ich seit einigen Jahren die Göttinger Nachrichten erhalte. Ich erlaube mir, dies beizufügen um zu verhüten, dass ich irrtümlich ein zweites Mal gewählt werde." (GAA Pers 19, 501)

On November 28, 1933, after several inquiries about Einstein at other academies and also at the German Embassy in Washington, the Academy's Secretary informed the Curator of Göttingen University that Einstein had been deleted from the list of external members of the Academy (GAA Pers 66, 18–23). Letters of reconciliation were sent out to the excluded members in August 1945 and again in December 1946 (GAA Pers 6, 10, 36). To these letters, Einstein did not respond.

[137]"Es ist objektiv schade, wenn sich zwei wirkliche Kerle, die sich aus dieser schäbigen Welt etwas herausgearbeitet haben, nicht gegenseitig zur Freude gereichen." Einstein to Hilbert, 20 December 1915, CPAE8, Doc. 167.

gave a lecture to the Göttingen Mathematical Society on "invariant theory and the general energy theorem."[138] And since his final communication was probably in press by mid-February, we may assume that he had found his new energy expression by that time and that his talk of January 25 presented a discussion of the energy concept as contained in the published version of his communication.

In the proofs, as discussed above, Hilbert had derived his energy concept by looking at the expression $\sqrt{g}J^{(p)}$ where $J^{(p)}$ was the "polarisation" of the invariant H with respect to the metric $g^{\mu\nu}$. The result had been an "energy form"

$$E = e_s p^s + e_s^l p_l^s \tag{40}$$

which could also be written in terms of the gravitational field equations and a pure divergence term as

$$E = -\left[\sqrt{g}H\right]_{\mu\nu} p^{\mu\nu} + \left(D^{(h)}_{(h=p)} - D^{(p)}\right). \tag{41}$$

The term e_s had been identified as a total derivative of $\sqrt{g}H$ taking into account only the dependence of the world function H on the metric and its derivatives.

In the published paper, Hilbert now formed a "polarisation" of $\sqrt{g}H$ with respect to all variables, the metric components and the electromagnetic potential. In the notation of his Theorem II, he considered the expression $P(\sqrt{g}H) = P_g(\sqrt{g}H) + P_q(\sqrt{g}H)$. Now Hilbert's aim was to derive an expression which only depended on the arbitrary vector p^s linearly, and which had a vanishing coordinate divergence. He showed that $P(\sqrt{g}H)$ could be written as

$$P(\sqrt{g}H) = \sum_{\mu\nu} \left[\sqrt{g}H\right]_{\mu\nu} p^{\mu\nu} + \sum_k \left[\sqrt{g}H\right]_k p_k + \frac{\partial\sqrt{g}(a^l + b^l + c^l + d^l)}{\partial w_l} \tag{42}$$

[138] JDM 25 (1917), p. 31.

where

$$a^l = \sum_{\mu,\nu,k} \frac{\partial H}{\partial g^{\mu\nu}_{kl}} \left\{ p^{\mu\nu}_k + \sum_\rho \left(\begin{Bmatrix} k\rho \\ \mu \end{Bmatrix} p^{\rho\nu} + \begin{Bmatrix} k\rho \\ \nu \end{Bmatrix} p^{\rho\mu} \right) \right\}, \tag{43}$$

$$b^l = \sum_{\mu,\nu,\rho,k} \left(\frac{\partial H}{\partial g^{\mu\nu}_l} - \frac{\partial}{\partial w_k} \frac{\partial H}{\partial g^{\mu\nu}_{lk}} - \frac{\partial H}{\partial g^{\rho\nu}_{lk}} \begin{Bmatrix} k\mu \\ \rho \end{Bmatrix} - \frac{\partial H}{\partial g^{\mu\rho}_{lk}} \begin{Bmatrix} k\nu \\ \rho \end{Bmatrix} \right) p^{\mu\nu}, \tag{44}$$

$$c^l = \sum_{kl} \frac{\partial H}{\partial q_{kl}} p_k, \tag{45}$$

$$d^l = \frac{1}{2\sqrt{g}} \sum_{k,s} \frac{\partial}{\partial w_k} \left\{ \left(\frac{\partial \sqrt{g} H}{\partial q_{lk}} - \frac{\partial \sqrt{g} H}{\partial q_{kl}} \right) p^s q_s \right\} \tag{46}$$

were shown to be contravariant vectors and constructed in such a way that terms which contained derivatives of p^s were only contained in the first two terms on the right hand side of eq. (42) which vanish on stipulation of the field equations. Invoking his theorem II, Hilbert now showed that $P(\sqrt{g}H)$ could alternatively be written as

$$P(\sqrt{g}H) = \sum_s \frac{\partial \sqrt{g} H p^s}{\partial w_s}. \tag{47}$$

Hence the expression

$$e^l = Hp^l - a^l - b^l - c^l - d^l \tag{48}$$

depended linearly on p^s and transformed as a vector under general coordinate transformations. Invoking the field equations, its covariant divergence

$$\sum_l \frac{\partial \sqrt{g} e^l}{\partial w_l} = 0 \tag{49}$$

vanishes identically in any coordinate system.[139]

Moreover, his other considerations regarding the energy-momentum tensor remained valid since the electromagnetic part of the energy followed from his new energy expression as

$$Lp^l - \sum_k \frac{\partial L}{\partial q_{kl}} p_k - \frac{1}{2\sqrt{g}} \sum_{k,s} \frac{\partial}{\partial w_k} \left\{ \left(\frac{\partial \sqrt{g} L}{\partial q_{lk}} - \frac{\partial \sqrt{g} L}{\partial q_{kl}} \right) p^s q_s \right\}, \tag{50}$$

[139] [Hilbert 1915, p. 402].

as is readily seen by looking at the definition of e^l in eq. (48) and of a^l, b^l, c^l, d^l in eqs. (43)–(46) and taking into account that the electromagnetic part L does not depend on the derivatives of the metric. Using that L depended, in fact, only on the fields M_{kl} and, invoking the generalized Maxwell equations, Hilbert transformed this expression to

$$\sum_{s,k}\left(L\delta_s^l - \frac{\partial L}{\partial M_{lk}}M_{sk} - \frac{\partial L}{\partial q_l}q_s\right)p^s. \tag{51}$$

And since eq. (34) which followed directly from Hilbert's second theorem remained valid, it followed that the electromagnetic part of his new energy concept again yielded Mie's energy-momentum tensor. Hence the conclusions about Mie's theory as a special case of Hilbert's theory also remained valid.

But the new energy vector was identically conserved, by virtue of eq. (49), for any system of coordinates. Hence the stipulation of its validity by an extra axiom did not add anything to the theory, its validity was already implied by the first two axioms, and hence the third axiom of the proofs was no longer independent from the first two axioms and had to be dropped. This result was in agreement with Einstein's conclusions of his final November memoir.

Hilbert might also have thought about the relation of his version of the field equations given as variational derivatives of an invariant and Einstein's final field equations which were given in terms of the Ricci tensor and the Riemann curvature scalar. In the published version, he added a remark to the effect that for the splitting of H into $H = K + L$, with K representing the Riemann scalar and L the electromagnetic part depending only on q_s, q_{sl} and $g^{\mu\nu}$, the first term on the left hand side of the field equations (35) assumed the explicit form

$$[\sqrt{g}K]_{\mu\nu} = \sqrt{g}(K_{\mu\nu} - \frac{1}{2}Kg_{\mu\nu}), \tag{52}$$

and justified this assertion that it followed "without calculation" from the fact that $K_{\mu\nu}$ and $g^{\mu\nu}$ are the only tensors of second rank and K the only invariant which can be formed out of the $g^{\mu\nu}$ and its first and second derivatives.[140] The argument may not follow so easily without calculation but is nevertheless true if it is understood that the second derivatives of the metric tensor enter only linearly and if the condition is taken into account that the combination

[140][Hilbert 1915, p. 405].

of $K_{\mu\nu}$ and $g^{\mu\nu}K$ has to satisfy the contracted Bianchi identity (24) derived in Hilbert's Theorem III.[141]

With the representation of his field equations in terms of the Ricci tensor, Hilbert also had to comment on the relation of his own field equations to those published by Einstein in his November equations. The comparison was not so easy because, for one, the source term of the field equations in Hilbert's and in Einstein's theory were not necessarily identifiable. In Einstein's theory it was an unspecified "energy tensor of matter," exemplified either by an incoherent, pressureless flow of particles (dust) or by the energy-momentum tensor of Maxwellian electrodynamics. For Hilbert it was the electromagnetic part of his energy vector e^l. With Hilbert's splitting of the "world function" H into a gravitational part K and an electromagnetic part L and with his result that, as a consequence of general covariance, L could only depend on the electromagnetic potential q_s and on its antisymmetrized derivatives M_{sl}, there is hence some freedom to determine L and hence to determine the energy-momentum tensor of matter in Mie's sense. Second, Hilbert never set the determinant of $\sqrt{g}$ equal to 1 as Einstein was still doing "in the usual manner"[142] even in his fourth November communication where such a specialization was no longer necessary for the equations to hold.[143] And the presentation of what amounted in fact to three different versions of field equations by Einstein within four weeks may have made it difficult to decide unambiguously what Einstein's new field equations really were. In fact, what may seem from hindsight as three distinct and alternative versions

[141]It is also possible that Hilbert took the factor $-1/2$ in eq. (52) for the trace term $g^{\mu\nu}K$ by looking at the field equations of Einstein's final November equation, as Corry, Renn and Stachel suggest ([Corry, Renn and Stachel 1997, p. 1272]). However, Einstein added the trace term to the matter tensor $T_{\mu\nu}$ and the equivalence of these two forms rests not only on the trivial equivalence of putting the trace term to the left or right hand side of the field equation but also on the perhaps more problematic identification of Einstein's $T_{\mu\nu}$ with Hilbert's $\frac{1}{\sqrt{g}}\frac{\partial\sqrt{g}L}{\partial g^{\mu\nu}}$.

[142]"in der gewohnten Weise" [Einstein 1915a, p.845].

[143]In a postcard to Schwarzschild, dated 26 February, 1916, Hilbert explicitly pointed out this difference to Einstein's theory: "Ich möchte Sie nur darauf aufmerksam machen, dass die Forderung Determinante $g = |g_{\mu\nu}| = 1$ ganz willkürlich und durchaus überflüssig ist. In meiner Theorie kommt dieselbe garnicht in Frage. Vgl. meine Gl. (21) S. 10." (SUB Cod. Ms. Schwarzschild 331/8.) The reference is to the field equations (35) which in the proofs were eq. (26). When Klein undertook his systematic comparison of Einstein's and Hilbert's energy expressions in 1918, he also complained about the changing treatment of the $\sqrt{g}$ factor in Einstein's papers, [Hilbert and Klein 1985, p. 142].

of field equations was presented by Einstein as essentially the same set of equations plus certain hypotheses about the choice of coordinates and about the constitution of matter, or modifications of the vacuum field equations in the presence of matter. Nevertheless, Hilbert conjectured that the final equations advanced by Einstein were equivalent to his own if K is taken to be the Riemann scalar:

> The resulting differential equations of gravitation are, it seems to me, in agreement with the broad theory of general relativity established by Einstein in his later papers.[144]

And to this passage he added a reference to all four November communications by Einstein, including the final one, submitted to the Berlin Academy on November 25 and published a week later on December 2, 1915. With these remarks Hilbert's note was prepared for press, by mid-February Hilbert received offprints of his paper and, by the end of March, the last 1915 issue of the *Nachrichten* was eventually published.

6 "A certain resentment"

With and after the publication of their respective notes, neither Einstein nor Hilbert themselves publicly ever entered into a dispute of priority. Nevertheless, during the hectic period of November some tension between Hilbert and Einstein had arisen as is clear already from the tone of the correspondence of that time and, in particular, from Einstein's explicit offer of reconciliation in his response of December 20 to Hilbert's informing him about his election as a corresponding member of the Göttingen Academy. In this letter Einstein wrote:

> There was a certain resentment between us, the cause of which I do not want to analyse. I have fought against the associated feeling of bitterness, and with complete success. I again think of you with unmixed friendliness, and I ask you to try to think of me in the same way.[145]

[144]"Die so zu Stande kommenden Differentialgleichungen der Gravitation sind, wie mir scheint, mit der von Einstein in seinen späteren Abhandlungen aufgestellten großzügigen Theorie der allgemeinen Relativität im Einklang." [Hilbert 1915, p. 405].

[145]"Es ist zwischen uns eine gewisse Verstimmung gewesen, deren Ursache ich nicht

In fact, Einstein had voiced bitter feelings against Hilbert in a letter to Heinrich Zangger, one of his closest friends, written on November 26, one day after Einstein had presented his final field equations, representing the completion of his General Theory of Relativity to the Berlin Academy. In this letter to Zangger of November 26, Einstein had accused Hilbert of the "nostrification" of his results.[146]

As John Earman and Clark Glymour in their account of the November 1915 episode emphasized, "questions about the priority of discoveries are often among the least interesting and least important issues in the history of science."[147] Nevertheless, for the reconstruction of conceptual innovation in the natural sciences, they may have a heuristic value in the identification of new insights, and Hilbert's first communication on the foundation of physics has often been commented on in the literature in this respect.[148]

Both Hilbert and Einstein saw their achievements of November 1915 as the culmination of year-long efforts of scientific research along their respective research programs. This may account for a certain irritability on both sides. But since Einstein's and Hilbert's research programs were by no means identical, the question arises as to the identification of those issues where both scientists felt that they had to secure their claims of priority. Einstein's resentment against Hilbert cannot have been induced by Hilbert's publication of the field equations in the explicit form in terms of the Ricci tensor and the Riemann scalar, nor in the establishment of full, unrestricted general covariance. His accusation of nostrification was expressed at the time of Hilbert's first proofs which did not yet contain the explicit form of the field equations and which still postulated a restriction of the covariance by the additional four space-time equations postulated by the third axiom in the proofs. And this third axiom was only dropped in the published version when Hilbert had found his new energy vector. Hence, one may also question whether even Einstein's offer of reconciliation of December 20 was a reaction

analysieren will. Gegen das damit verbundene Gefühl der Bitterkeit habe ich gekämpft, und zwar mit vollständigem Erfolge. Ich gedenke Ihrer wieder in ungetrübter Freundlichkeit, und bitte Sie, dasselbe bei mir zu versuchen." Einstein to Hilbert, 20 December 1915, CPAE8, Doc. 167.

[146]This letter is the subject of [Medicus 1984] and is, in fact, undated. It has been dated by context to November 26 in Volume 8 of the *Collected Papers of Albert Einstein*. To be precise, Einstein does not mention Hilbert by name but it seems beyond doubt that Hilbert was meant.

[147][Earman and Glymour 1978, p. 291].

[148][Guth 1970], [Mehra 1974] seem to have been the first to raise the issue of priority.

to Hilbert's revisions of his paper.

It seems more probable that Einstein regarded Hilbert's axiomatic interpretation of his theory as a "nostrification" and that he objected to Hilbert's treating his gravitation theory as a mere mathematical preliminary to his own theory. In fact, in his first axiom Hilbert had postulated that the "world function" H would depend on the components of the metric tensor $g^{\mu\nu}$, its first and second derivatives, as well as on the electromagnetic potential q_s and its derivatives. And both quantities had been introduced in the preceding paragraph on the same footing as the "quantities which characterize the processes in the world,"[149] namely

> 1) the ten gravitation potentials $g_{\mu\nu}$ $(\mu, \nu = 1, 2, 3, 4)$ with symmetric tensor character with respect to an arbitrary transformation of the world parameter w_s;
> 2) the four electrodynamic potentials q_s with vector character in the same sense.[150]

From hindsight, it was the introduction of the metric tensor which implied the most radical break with classical space-time concepts by providing both the chronogeometrical and the inertio-gravitational structures and which thus represented *the* "crucial step in the development of general relativity."[151] But already at the time, Einstein had a clear understanding of the revolutionary and innovative conceptual implications of his general theory of relativity which, at that time, had also made him an outsider in the field of gravitation theory.[152] Hilbert had acknowledged "the tremendous research problems of Einstein and his perspicaciously devised methods for their solution"[153] in the first paragraph of his note. But, in the published version, he explicitly gave credit to Einstein for the introduction of the metric tensor, adding that those ten gravitational potentials were "first introduced by Einstein."[154]

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[149] "Die das Geschehen in w_s charakterisierenden Größen ..." Proofs, p. 1, [Hilbert 1915, p. 395].

[150] "1) die zehn Gravitationspotentiale $g_{\mu\nu}$ $(\mu, \nu = 1, 2, 3, 4)$ mit symmetrischem Tensorcharakter gegenüber einer beliebigen Transformation der Weltparameter w_s; 2) die vier elektrodynamischen Potentiale q_s mit Vektorcharakter im selben Sinne." Proofs, p. 1

[151] [Stachel 1995, p. 293]; see also [Stachel 1994].

[152] For a contemporary review of gravitational theories, see [Abraham 1915].

[153] "[...] die gewaltigen Problemstellungen von Einstein sowie dessen scharfsinnige zu ihrer Lösung ersonnenen Methoden," proofs, p. 1, [Hilbert 1915, p. 395].

[154] "[...] die zehn zuerst von Einstein eingeführten Gravitationspotentiale [...]," [Hilbert 1915, p. 395].

While Hilbert thus gave credit to the conceptual justification of Einstein's metric theory of gravitation, he nevertheless claimed that he had independently derived the field equations of General Relativity from a variational principle. So, of course, did Einstein who did not refer to Hilbert in his final communication. To be sure, since John Norton's 1984 account of Einstein's route towards general relativity who argued that "Einstein's final steps were self-contained"[155] no serious argument was ever advanced disputing Einstein's independence in deriving his field equations of November 25. Nevertheless, since Hilbert's published note only contained the date of presentation to the Academy, November 20, predating Einstein's fourth communication, which was presented to the Berlin Academy on November 25, by five days, some commentators have haphazardly conjectured that Einstein may have taken the trace term introduced in his final communication by looking at Hilbert's paper.[156] This conjecture can now be regarded as definitely refuted by the first proofs of Hilbert's paper.[157]

But the independence of Einstein's discovery was never a point of dispute between Einstein and Hilbert. Nor was the independence of Hilbert's derivation of the field equation ever disputed by Einstein. Hilbert claimed priority for the introduction of the Riemann scalar into the action principle and the derivation of the field equations from it,[158] and Einstein admitted publicly that Hilbert (and Lorentz) had succeeded in giving the equations of general relativity a particularly lucid form by deriving them from a single variational

[155][Norton 1984, p. 314].

[156]See, e.g., [Fölsing 1993, p. 421].

[157][Corry, Renn and Stachel 1997].

[158]Thus, Hilbert apparently had objected to the presentation in a paper by Herglotz on the geometric implications of the introduction of the Riemann tensor into gravitation theory published in early 1917 ([Herglotz 1916]). In a defensive response, Herglotz admitted that he should have pointed out that the tensor $K_{\mu\nu} - \frac{1}{2}g_{\mu\nu}K$ first appeared naturally as a variation of $\int K\sqrt{g}dw$ in Hilbert's paper. ("Ich hätte freilich auf das erstmalige natürliche Auftreten des Tensors $K_{\mu\nu} - \frac{1}{2}g_{\mu\nu}K$ als Variation von $\int K\sqrt{g}dw$ in Ihren 'Grundlagen' besonders hinweisen sollen." Herglotz to Hilbert, undated, SUB Cod. Ms. Hilbert 147.)

And in a draft of a letter to Weyl, dated 22 April 1918, written after he had read the proofs of the first edition of Weyl's "Raum—Zeit—Materie" Hilbert also objected to being slighted in Weyl's exposition. In this letter again "in particular the use of the Riemannian curvature [scalar] in the Hamiltonian integral" ("insbesondere die Verwendung der Riemannschen Krümmung unter dem Hamiltonschen Integral") was claimed as one of his original contributions. SUB Cod. Ms. Hilbert 457/17.

principle.[159]

Hilbert's irritation in November 1915, I would like to suggest, referred to Einstein's "Nachtrag" to his first November communication, presented to the Berlin Academy on November 11.[160] In this "Nachtrag" Einstein had advanced generally covariant field equations, equating the Ricci tensor to the energy-momentum tensor of matter, and justified these equations by the hypothesis that the trace of the energy-momentum tensor of matter vanish. This was a necessary consequence of these field equations if one wanted to fix the coordinates such that $\sqrt{-g}$ be a constant, and this coordinate condition had to be imposed in order to recover the field equations of Einstein's first November communication. Since the trace of the energy-momentum tensor vanishes for the electromagnetic energy-momentum tensor but not necessarily for other tensors, e.g. not for the energy-momentum tensor of incoherent pressureless dust, the field equations hence implied a hypothesis about the constitution of matter. As Einstein said:

> There are indeed not a few who hope to be able to reduce matter to purely electromagnetic processes. These processes, however, would have to be governed by a theory which generalizes Maxwell's electrodynamics to a completed theory.[161]

Clearly, Hilbert must have understood this as a reference to Mie's electrodynamic field theory of matter. And Einstein's claim that the field equations implied a hypothesis about the constitution of matter just touched on the "Leitmotiv" of Hilbert's own theory. Indeed, as was discussed above, Hilbert interpreted his first theorem about the implied existence of four identities between the 14 field equations and their derivatives, in the sense that the electrodynamic phenomena are effects of gravitation. And it is well possible that Hilbert had informed Einstein about this characteristic of his theory in that "friendly letter" which, unfortunately lost, he had sent Einstein in response to Einstein's initial correspondence of November 7.

But the subsequent introduction of the trace term in Einstein's field equations of November 25, implied not only that energy-momentum conservation

[159]"In letzter Zeit ist es H.A. Lorentz und D. Hilbert gelungen, der allgemeinen Relativitätstheorie dadurch eine besonders übersichtliche Gestalt zu geben, daß sie deren Gleichungen aus einem einzigen Variationsprinzipe ableiten." [Einstein 1916b, p. 1111].

[160][Einstein 1915c].

[161]"Es gibt sogar nicht wenige, die hoffen, die Materie auf rein elektromagnetische Vorgänge reduzieren zu können, die allerdings einer gegenüber Maxwell's Elektrodynamik vervollständigten Theorie gemäß vor sich gehen würden." [Einstein 1915c, p. 799].

no longer imposed a restriction on the admissible coordinates. It also made the hypothesis of the "Nachtrag" superfluous, a consequence which Einstein explicitly pointed out:

> On the other hand the postulate of general relativity cannot disclose anything about the other phenomena of nature that would not already follow from special relativity. My former opinion, expressed recently in this forum, was erroneous.[162]

With Einstein's retreat regarding his claim about implications of his gravitational field equations for an inherent electromagnetic constitution of matter, the danger of a priority problem for Hilbert, as far as Einstein was concerned, had vanished.

7 Concluding remarks

The substantial lasting innovation of Hilbert's first note on the foundation of physics was the foundation of Einstein's general theory of relativity on an invariant variational principle as an equivalent representation of the gravitational and electromagnetic field equations and the mathematical elaboration of some consequences which follow alone from the invariance of the action integral with respect to arbitrary transformations of coordinates. These insights include the discovery of a special case of Noether's second theorem, the derivation of the generalized contracted Bianchi identities from the variational principle in his Theorem II, as well as the introduction of the Riemann curvature scalar into the variational integral. Other innovative features of his note have not stood the test of time. Among these are his ideas on a unified field theory of gravitation and electromagnetism and his energy vector.

With regard to the final establishment of a theory of unrestricted, general covariance and the interaction between Hilbert and Einstein in this matter, I should like to venture the following historical assessment of Hilbert's work. Hilbert's knowledge and understanding of the calculus of variations and of invariant theory readily put him into a position to fully grasp Einstein's

[162]"Dagegen vermag das allgemeine Relativitätspostulat uns nichts über das Wesen der übrigen Naturvorgänge zu offenbaren, was nicht schon die spezielle Relativitätstheorie gelehrt hätte. Meine in dieser Hinsicht neulich an dieser Stelle geäußerte Meinung war irrtümlich." [Einstein 1915a, p. 847].

gravitation theory of 1914 and the mathematical intricacies of the derivation of its field equations from a variational principle. This understanding and the subsequent axiomatic reinterpretation of Einstein's theory as well as Hilbert's way of restricting general covariance in order to guarantuee energy conservation and causality by means of a third, independent axiom in the first proofs of his note, posed, it seems to me, objectively a threat to the delicate, metastable balance between physical conceptions and their mathematical representation[163] of Einstein's *Entwurf* theory, as achieved by his 1914 exposition of the *Formale Grundlage*. And whatever Hilbert later learnt from reading Einstein's final November communication while reviewing the proofs of his note, I should like to suggest that Hilbert's own justification of general covariance by means of the revised energy concept of the published paper, was based also on arguments of internal mathematical coherency which were independent from Einstein's considerations.

Hilbert's specific contributions to the history of general relativity as well as their limitations were conditioned by a vast knowledge of mathematics and a broad knowledge of contemporary theoretical physics, by the heuristics of Hilbert's axiomatic method of identifying basic assumptions and their respective implications and of looking for fundamental and intricate mathematical questions associated with physics, as well as by Hilbert's belief in the unity of the mathematical sciences and in the feasibility of turning physics into a mathematical discipline based on an axiomatic foundation.

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[163][Renn and Sauer 1998].

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Abbreviations

CPAE: The Collected Papers of Albert Einstein
GAA: Archiv der Akademie der Wissenschaften zu Göttingen.
GM: Geschäftliche Mitteilungen der Königlichen Gesellschaft der Wissenschaften in Göttingen
JDM: Jahresbericht der Deutschen Mathematikervereinigung. 2. Abteilung
SUB: Niedersächsische Staats- und Universitätsbibliothek Göttingen, Handschriftenabteilung.

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