Neuchatel #2, a talk at Neuchatel GST Summer School, Monday September 7, 1998
Submitted to the Proceedings, Eric Schwartz, ed., Saturday December 19, 1998

# Complex Dynamical Systems

by

Ralph H. Abraham

Professor of Mathematics, University of California at Santa Cruz
Visual Math Institute, www.vismath.org
POB 7920, Santa Cruz, CA 95061

Dedicated to Anatol Rapoport

**Abstract**
Complex dynamical systems theory and system dynamics diverged at some point in the recent past, and should reunite. This is a concise introduction to the basic concepts of complex dynamical systems, in the context of the new mathematical theories of chaos and bifurcation.

**Contents**

## 1. Introduction

Dynamical systems theory begins with Newton in the 17th century and took a radical turn with Poincaré a century ago. During this century, the community of scientists using dynamical systems to model natural systems discovered a technique of combining simple systems into networks, and this movement evolved, through cybernetics and general systems theory, into the branch of mathematics now known as *complex dynamical systems theory*, or nearly equivalently, as *systems dynamics*.

Dynamical systems occur in four different flavors:

- flows, or continuous-time, autonomous dynamical systems,
- cascades, or discrete-time, reversible dynamical systems,
- iterations, or discrete-time, irreversible dynamical systems, and
- shifts, or symbolic dynamical systems.

Flows have been used since Newton, are the best understood models, and are most often used by scientists. Cascades and iterations were introduced by Poincaré, and symbolic dynamics followed soon after. Here, for the sake of simplicity, we speak only of flows.

Our intention is to present, without rigorous definitions, a minimal lexicon for dynamical literacy. For those interested in pursuing the ideas further, the references should be consulted. A more extensive and annotated bibliography may be found at the website of the Visual Math Institute, www.vismath.org.

## 2. Dynamical systems

A *flow*, or continuous dynamical system, is generated by a vectorfield on a state space. The *state space* may be a Euclidean space or a smooth manifold of any dimension, finite or infinite. For beginners, it is most helpful to think in terms of Euclidean spaces of dimension one, two, or three. The *generator* of the system is a field of vectors which may be regarded as giving the required vector velocity, at any given point, which the trajectories of the system must have.
A *trajectory* is a curve in the state space having at each point the required velocity vector. The *flow*, then, is the set of all points in the state space, moving along the trajectories like a fluid.

When followed for a long time, most trajectories end up in a dynamical equilibrium called an *attractor*. Attractors may be classified in three or more categories, such as fixed, periodic, or chaotic. Normally there are many attractors. Fixing attention on one attractor, we might mark each and every state which ends up at this attractor in the long run. The set of states so marked comprise the *basin* of the attractor. Basins may be fat, like lakes, or thin, like meandering rivers.

Distinct basins are bounded and separated by *basin boundaries*, also called *separatrices*. These may be *thin*, like points in dimension one, curves in dimension two, surfaces in dimension three, and so on. Very often, however, they are *thick*, that is to say, fractal.

The most useful image of a dynamical system is its *portrait*: the state space divided into basins by separatrices, with one attractor indicated in each basin.

## 3. Dynamical schemes

Dynamical systems are much employed in the mathematical modeling and computer simulation of the simpler systems found in nature. Such models usually have some *control parameters*, variable coefficients for example, which are tuned by the modeler so as to obtain the best fit between some given experimental data and the simulated data output by the model. A model with controls is called a *dynamical scheme*, or alternately, a parameterized family of dynamical systems.

When the control parameters of a scheme are fixed, we then have a dynamical system, one member of the parametrized family. This system may be visualized as a portrait. And if the controls are then moved, the portrait will be changed. When the controls are moved smoothly and gradually, the portrait may be seen to also change smoothly and gradually. Sometimes, however, the portrait undergoes a radical change even when the controls are moved very gently. Such an event is called a *bifurcation*.

Bifurcations are certainly the most important features of a scheme, and locating them is a difficult job for the experimentalist. One might begin a study of bifurcations by looking at some exemplary cases, and most elementary texts do just this. The simple examples fall into three categories:

- *subtle bifurcations*, in which the change is not immediately striking,
- *catastrophic bifurcations*, in which a basin suddenly appears or disappears, and
- *explosive bifurcations*, in which an attractor suddenly expands or contracts.

## 4. Complex dynamical schemes

The schemes described above, and those usually found in elementary texts, are *simple schemes*. Their state spaces have low dimension, and they have just a few control parameters. They are suitable for modeling only the simplest natural systems, such as a simple pendulum. More complex natural systems require model schemes made by combining several simple schemes in a network. These are called *complex dynamical schemes*.

One begins with a *directed graph*, that is, a diagram with blank boxes, *nodes*, connected by arrows, or *connections*. The nodes, corresponding to subsystems of the natural system, must be filled in with specific simple schemes. The connections must be specified by coupling functions, which enslave some controls (*inputs*) of a target scheme to the states (*outputs*) of a source scheme.

After being connected in this way, some of the controls of the node schemes are enslaved, and are thus no longer control parameters. Other node controls remain free. Thus, the fully connected complex scheme is still a dynamical scheme. The meaning of *complex* in this context refers to the means of construction of the model, as a system of subsystems.

Neural nets are complex dynamical schemes. So are most models in mathematical biology, ecology, atmospheric science, and so on. The evolving experience with massively complex schemes has led to an idea, called *connectionism*, that the network is more important than the choice of models for the nodes.

## 5. Software

Years ago, we taught our students to build models using programming languages such as BASIC, PASCAL, C, and so on. Special simulation languages such as DYNAMO and STELLA came along and greatly advanced the art of complex dynamical modeling. Then came the fabulous environments for symbolic manipulation and general math problem solving such as MACSYMA, MAPLE, MATHEMATICA, MATLAB, and the like. Today it is relatively simple to build a complex dynamical scheme to model scientific data. The larger problem is to understand the model, and this is where the evolving theory of complex dynamics (systems dynamics) comes to our aid.

## 6. Conclusion

Today we live in a world troubled by many large problems. One difficulty in facing these problems is their sheer complexity. We have, collectively, a complexity horizon. Systems within the horizon we can understand; those outside are beyond our ken. And the *world problematique* is well over the horizon.

To a certain extent, our problem consists of the rejection of the solution. For we now have advanced methods for the mathematical modeling and computer simulation of complex dynamical systems which can significantly expand our complexity horizon. These new methods are greatly underutilized, as our society is handicapped by a mass epidemic of math anxiety and math avoidance syndrome.

A challenge for the world community of general systems thinkers is to influence the educational systems of all nations so as to promote dynamical literacy, and the systems approach to understanding global problems.

## Bibliography

Abraham, Ralph H., 1991. CDS, *Complex Dynamics*, Aerial Press, Santa Cruz.

Abraham, Ralph H., and Christopher D. Shaw, 1992. DGB2, *Dynamics, The Geometry of Behavior, Second edition*, Addison-Wesley, Reading MA.

Abraham, Ralph H., Laura Gardini, and Christian Mira, 1997. JPX. *Chaos in Discrete Dynamical Systems*, Springer-Verlag, New York.

Laszlo, Ervin, 1991. *The Age of Bifurcation*, Gordon and Breach, Philadelphia.

Von Bertalanffy, Ludwig, 1972. The history and development of general systems theory, in: *The Relevance of General Systems Theory; papers presented to Ludwig von Bertalanffy on his seventieth birthday.* Edited by Ervin Laszlo, Braziller, New York.